NATURAL COMPACTIFICATIONS OF LATTICES

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KEY OF SYMBOLS

X ..... a topological lattice (2.8 (iii))

$X_D(a) = \{x \in X: x \leq a\}$ and $X_I(a) = \{x \in X: a \leq x\}$

Γ_D ..... (2.3 (ii)) the collection of all closed ideals of X

Γ_I ..... (2.3 (ii)) the collection of all closed dual ideals of X

$\Gamma = \Gamma_D \cup \Gamma_I$

T_D ..... (2.3 (iii)) the collection of all principal ideals of X together with X

T_I ..... (2.3 (iii)) the collection of all principal dual ideals of X together with X

$T = T_D \cup T_I$.. (4.1 (iii)) is the usual subbase for X

Σ ..... an arbitrary admissible subbase for X (4.2 (i))

$\lambda_\Sigma X$ ..... (2.15) the superextension of X with respect to Σ

S^+ (or sometimes S^* to avoid confusion) ... (2.14) the set of all maximal linked systems of Σ which contain S

Σ^+ (or sometimes Σ^*) $= \{S^+: S \in \Sigma\}$. (2.14)

e_Σ ..... (2.16 (v)) the natural embedding of X into $\lambda_\Sigma X$

$\beta_\Sigma X$ ..... (2.17) the topological closure of $e_\Sigma[X]$ in $\lambda_\Sigma X$ (called the de Groot compactification of X with respect to Σ)

$\Psi^\cap$ ..... the collection of arbitrary intersections of Ψ

2^X ..... the power set of X

$D(A) = \cap\{X_D(a): A \subseteq X_D(a)\}$ where $A \in 2^X$ (5.3)

$\tilde{X}$ ..... (5.2) the completion of X by cuts

$d: X \to \tilde{X}$ ..... (5.4) the map defined by $a \to D(\{a\})$

$\tilde{T} = \{\tilde{X}_D(d(a)): a \in X\} \cup \{\tilde{X}_I(d(a)): a \in X\} \cup \{\tilde{X}\}$ (5.7)

ωX ..... the Alexandroff one-point compactification of X

$Z^1 = Z^{1a} \cap Z^{1b}$ (6.1)

$Z^2 = Z^{2a} \cap Z^{2b}$ (6.1)

$\tilde{T} = \{Y_D(i(a)): a \in X\} \cup \{Y_I(i(a)): a \in X\} \cup \{Y\}$ (where $Y = i[X]^1$) (6.6)

$\Sigma^{(i)}$ ..... (6.15)

1. INTRODUCTION

Armed only with the definition of compactification, and asked to compactify the open disc in the plane, a novice topologist would most likely simply add its boundary. However, among the "standard" methods which have been developed for obtaining compactifications of spaces (Stone-Čech, Alexandroff, Freudenthal, Wallman, etc.) not one yields this seemingly "natural" compactification.[1] The Stone-Čech compactification (although it has nice mapping properties) adds so many points that the disc's "intrinsic character" is not preserved, and it is in some sense "lost" in the new space. At the other extreme, the Alexandroff compactification does not add enough. What we seek, then, is a method for producing compact extensions which yields "natural" compactifications in the sense that the closed interval [a, b] "naturally" compactifies the open interval (a, b), the extended real line "naturally" compactifies the rationals, the closed disc "naturally" compactifies the open disc, and the Hilbert cube "naturally" compactifies Hilbert space. In this dissertation we define and investigate such a method.

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[1] Frink [8] has developed a Wallman-type method for obtaining many compactifications for a given space. It has been shown by Alo and Shapiro [2] that the compactification of the open disc by the closed disc is of Wallman-type.

The method which we use is due to de Groot. For a given space, this method produces compact extensions (called superextensions) and compactifications (called de Groot compactifications) which are dependent upon both the space and a chosen subbase for its closed sets. Hence, in general, for each space there may be many superextensions and de Groot compactifications. (See Chapter 2 for details.) In particular, for locally compact Hausdorff spaces, it is possible (by astutely choosing the subbases) to obtain both the Stone-Čech and the Alexandroff compactifications as de Groot compactifications. Another aspect of the versatility of the de Groot compactifications is that they (unlike any of the "standard" compactifications mentioned above) are productive. Specifically, we will show in Chapter 3 that the topological product of any collection of de Groot compactifications (respectively, superextensions) of a family of spaces is homeomorphic with a natural de Groot compactification (respectively, superextension) of the topological product of the family.

In order to obtain the desired "natural" compactifications, we consider only topological products of topological lattices (i.e., lattices with the interval topology) and restrict our attention, when dealing with lattices, to those superextensions which arise with respect to certain "admissible" subbases. This restriction is not overly severe since, as can be seen in Chapter 8, for a given topological lattice there are, in general, many "lattice superextensions" (i.e., superextensions with respect to admissible subbases). In Chapter 4 we will

show how the order of the original lattice can be extended to a partial ordering of any lattice superextension in such a way that the superextension becomes a complete lattice (in the interval topology). Thus each lattice superextension of a topological lattice turns out to be not only a topological compactification, but also a lattice completion--hence a very "natural" type of compactification.

In Chapter 5 we consider the algebraic structure of a topological lattice X and show that its completion by cuts can be realized as a particular lattice superextension. In Chapter 6 we attempt to algebraically characterize other lattice superextensions of X. This is done using the notion of topological lattice completions of X, these being certain complete lattices containing X as a sublattice. We show that all lattice superextensions of X are in fact topological lattice completions of X and that those topological lattice completions of X satisfying a particular "density" condition are lattice superextensions of X.

In [12] Kaufman considers (totally) orderable compactifications for totally ordered spaces X; i.e., compactifications Y of X for which the given topology on Y is the same as the order (interval) topology on Y. Among other things, he establishes the existence of a "largest" and a "smallest" orderable compactification for each totally ordered space. In Chapter 7 we consider the notion of "domination" for lattice superextensions of topological lattices and generalize Kaufman's results to these compactifications of lattices. Hence each topological lattice

has a lattice compactification which is maximal relative to mapping properties (analogous to the Stone-Čech compactification for more general spaces) and a lattice compactification which is minimal relative to mapping properties (analogous to the Alexandroff compactification for locally compact Hausdorff spaces).

Chapter 8 consists of examples of lattice superextensions of various kinds of lattices. Also, examples are given which show the necessity of certain restrictions which are made in the text.

2. PRELIMINARIES

The purpose of this chapter is to state most of the basic definitions, propositions, and theorems which will be needed in the following chapters. Algebraic terms which are not specifically defined here can be found in Birkhoff [5] and topological terms in Kelley [13].

DEFINITION 2.1. (i) A set X is called partially ordered if there is a binary relation $\leq$ on X which is reflexive, antisymmetric, and transitive.

(ii) If X is a partially ordered set with more than one element in which each pair of elements a and b have a supremum (denoted by $a \vee b$) and have an infimum (denoted by $a \wedge b$), then X is called a lattice.[1]

(iii) If X is a lattice and each subset C of X has a supremum in X and an infimum in X, then X is said to be complete.[2]

(iv) If X is a lattice and $a, b \in X$, then a and b are said to be comparable if either $a \leq b$ or $b \leq a$.

(v) If X is a lattice and $a, b \in X$, then $a < b$ will mean that $a \leq b$ and $a \neq b$.

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[1] We consider only non-trivial lattices in order to avoid the possibility that $\vee X = \wedge X$.

[2] Throughout we will use the standard conventions that for a lattice X, $\wedge\emptyset = \vee X$ and $\vee\emptyset = \wedge X$, and for a topological space X, $\cap\emptyset = X$.

(vi) X is said to be totally ordered if for every pair of elements x and y of X, x and y are comparable; i.e., $x \leq y$ or $y \leq x$.

To avoid confusion when more than one lattice is considered, the symbols "$\vee$, $\wedge$ and $\leq$" may be subscripted such as "$\vee_X$, $\wedge_X$, and $\leq_X$" to denote supremum in X, infimum in X, and order in X, respectively.

DEFINITION 2.2. Suppose that X and Y are lattices and $i: X \to Y$ is a function.

(i) The map i is called a lattice homomorphism whenever i satisfies the following: For each non-void subset C of X, if $\vee_X C$ exists in X, then $\vee_Y i[C]$ exists in Y and $i(\vee_X C) = \vee_Y i[C]$. Dually, if $\wedge_X C$ exists in X, then $\wedge_Y i[C]$ exists in Y and $i(\wedge_X C) = \wedge_Y i[C]$.

(ii) The map i is called a lattice inclusion and X is called a sublattice of Y if i is an injective lattice homomorphism.

(iii) The map i is called a lattice isomorphism and X is said to be lattice isomorphic with Y if i is a surjective lattice inclusion.

DEFINITION 2.3. Let X be a lattice and I a non-void subset of X.

(i) I is increasing if $a \in I$ whenever $a \in X$ and $i \leq a$ for some $i \in I$. I is decreasing if $a \in I$ whenever $a \in X$ and $a \leq i$ for some $i \in I$.

(ii) I is an ideal of X if I is decreasing and $a \vee b \in I$ for each pair $a, b \in I$. I is a dual ideal if I is increasing and $a \wedge b \in I$ for each pair $a, b \in I$.[3]

[3] We do not consider $\emptyset$ to be an ideal or dual ideal in order to simplify the statements of proofs.

(iii) I is a principal ideal of X if there is some fixed $a \in X$ such that $I = \{x \in X: x \leq a\}$. I is a principal dual ideal of X if there is some fixed $b \in X$ such that $I = \{x \in X: b \leq x\}$.

PROPOSITION 2.4. Let X be a lattice.

(i) Arbitrary intersections of ideals of X are either void or ideals of X, and arbitrary intersections of dual ideals of X are either void or dual ideals of X.

(ii) Finite intersections of ideals of X are ideals of X, and finite intersections of dual ideals of X are dual ideals of X.

(iii) If $\mathcal{F}$ is a finite collection of ideals of X such that $X \notin \mathcal{F}$, then $\cup \mathcal{F} \neq X$, and dually if $\mathcal{G}$ is a finite collection of dual ideals of X such that $X \notin \mathcal{G}$, then $\cup \mathcal{G} \neq X$.

PROOF. (i) Suppose ζ is a collection of ideals of X such that $\cap \zeta \neq \emptyset$. Let $I = \cap \zeta$. If $b \in I$ and $a \in X$ with $a \leq b$, then since $b \in E$ for each $E \in \zeta$ and each E is decreasing, $a \in E$ for each $E \in \zeta$. Hence $a \in I$; i.e., I is decreasing. Finally, if $a, b \in I$, then since $a, b \in E$ for each $E \in \zeta$ and each E is an ideal, it follows that $a \vee b \in E$ for each E, so that $a \vee b \in I$. By Definition 2.3 (ii), I is an ideal of X. A dual argument shows that the intersection of an arbitrary collection of dual ideals of X is either void or a dual ideal of X.

(ii) Suppose $I_1, I_2, \cdots, I_n$ are ideals of X. By definition, $I_j \neq \emptyset$ for each $j = 1,2,\cdots,n$ so there exists $x_j \in I_j$ for each j. It follows that $\wedge\{x_j: j = 1,2,\cdots,n\} \in I = \cap\{I_j: j = 1,2,\cdots,n\}$ since

each I_j is decreasing, and hence $I \neq \emptyset$. By (i) above, I is an ideal of X. A dual argument shows that the intersection of a finite collection of dual ideals of X is a dual ideal of X.

(iii) Since $X \notin \mathcal{F}$, for each $F \in \mathcal{F}$ there exists $x_F \in X - F$. Let $x = \vee\{x_F: F \in \mathcal{F}\}$. If $x \in \cup\mathcal{F}$, then $x \in F$ for some $F \in \mathcal{F}$, and since F is decreasing and $x_F \leq x$, we see that $x_F \in F$--a contradiction. Thus $x \in X - \cup\mathcal{F}$ and so $\cup\mathcal{F} \neq X$. A dual argument proves the other statement.

PROPOSITION 2.5. *If X and Y are lattices and i: X → Y is a lattice homomorphism, then i preserves the order of X.*

PROOF. Let i: X → Y be a lattice homomorphism and suppose $x, y \in X$ such that $x \leq_X y$. Thus $y = x \vee_X y$ and by the definition of lattice homomorphism $i(y) = i(x \vee_X y) = i(x) \vee_Y i(y)$; therefore $i(x) \leq_Y i(y)$ and so i preserves the order of X.

DEFINITION 2.6. Let X be a topological space and $A \subseteq X$. The topological closure of A in X will be denoted by $\overline{A}^X$ or $\overline{A}$ when no confusion seems likely.

DEFINITION 2.7. Suppose X and Y are topological spaces and f: X → Y is a function.

(i) f is called an *embedding* of X into Y if f is injective, continuous, and a closed map onto its image.

(ii) f is called a *dense embedding* of X into Y if f is an embedding and $\overline{f[X]}^Y = Y$.

(iii) f is called a homeomorphism and X is said to be homeomorphic with Y whenever f is a surjective embedding.

(iv) Y is called a compactification of X if Y is compact and f is a dense embedding.

DEFINITION 2.8. Suppose that X is a lattice.

(i) A closed interval of X is a subset A of X such that either A is a principal ideal of X, A is a principal dual ideal of X, or A is the non-void intersection of a principal ideal of X and a principal dual ideal of X. (In the latter case, there exist $a, b \in X$ such that $A = \{x \in X: a \leq x \leq b\}$.)

(ii) The interval topology on X is the topology on X which has all closed intervals of X as a subbase for the closed subsets of X.

(iii) A lattice X will be called a topological lattice when it is also considered as a topological space having the interval topology.

(iv) $\Sigma \subseteq 2^X$ will be called a subbase for X when Σ is a subbase for the closed sets of the topological lattice X and $X \in \Sigma$.

The following proposition is due to Frink [9].

PROPOSITION 2.9. If X is a topological lattice, then X is compact if and only if X is complete.

DEFINITION 2.10. Suppose that X and Y are topological lattices and $f: X \to Y$ is a function. f is called an iseomorphism and X is said to be iseomorphic with Y if f is both a lattice isomorphism and a homeomorphism.

The remainder of this chapter contains definitions and some fundamental results concerning superextensions of T_1-spaces. These results are due to de Groot, Jensen, and Verbeek [7]. It is within the framework of superextensions that we will study lattices in the following chapters.

DEFINITION 2.11. Suppose that X is a T_1-space and Σ is a subbase for the closed subsets of X.

(i) If A and B are subsets of X, then A and B are said to be screened by a subcollection Ψ of Σ and Ψ is called a screen for A and B if $\cup\Psi = X$ and no member of Ψ meets both A and B. Two elements x and y are screened by Ψ if $\{x\}$ and $\{y\}$ are screened by Ψ, and an element $x \in X$ and a subset $A \subseteq X$ are screened by Ψ if A and $\{x\}$ are screened by Ψ.

(ii) Σ is called a T_1-subbase for X if for each $x \in X$, $\cap\{S \in \Sigma: x \in S\} = \{x\}$, and whenever $x \in X$ and $S \in \Sigma$ with $x \notin S$, there is some $T \in \Sigma$ with $x \in T$ and $T \cap S = \emptyset$.

(iii) Σ is said to be weakly-normal if each pair of disjoint members of Σ can be screened by a finite collection from Σ.

(iv) A non-void subcollection $\mathcal{L}$ of Σ is said to be linked if each pair of elements of $\mathcal{L}$ has a non-void intersection; $\mathcal{L}$ is called a maximal linked system of Σ if $\mathcal{L}$ is maximal in Σ with respect to the property of being linked.

The next two propositions are found in [7]. Their proofs are included only to acquaint the reader with the terminology.

PROPOSITION 2.12. A space X is T_1 if and only if there is a T_1-subbase for the closed sets of X.

PROOF. Suppose that X is a T_1-space and let Σ be the collection of all closed sets of X. Clearly Σ is a subbase for the closed subsets of X. Let $x \in X$. Since X is T_1, $\{x\} \in \Sigma$, and it follows that $\{x\} \subseteq \bigcap\{S \in \Sigma: x \in S\} \subseteq \{x\}$; i.e., $\bigcap\{S \in \Sigma: x \in S\} = \{x\}$. If $x \in X$ and $S \in \Sigma$ with $x \notin S$, then clearly $\{x\} \in \Sigma$, $x \in \{x\}$, and $\{x\} \cap S = \emptyset$. Hence Σ is a T_1-subbase for X (2.11 (ii)).

Conversely, suppose that Σ is a T_1-subbase for X. Let $x \in X$. To see that $\{x\}$ is closed, let $y \in X$ such that $y \notin \{x\}$. Hence $y \notin \bigcap\{S \in \Sigma: x \in S\}$ since Σ is a T_1-subbase for X. Thus there is some closed set $S \in \Sigma$ with $\{x\} \subseteq S$ and $y \notin S$. It follows that $\{x\}$ is a closed subset of X.

PROPOSITION 2.13. If Σ is a subbase for the closed subsets of a T_1-space X, then Σ is a T_1-subbase for X if and only if for each $x \in X$, $\{S \in \Sigma: x \in S\}$ is a maximal linked system of Σ.

PROOF. Suppose that Σ is a T_1-subbase for X and $x \in X$. Let $\mathcal{H} = \{S \in \Sigma: x \in S\}$. Clearly $\mathcal{H}$ is a linked system of Σ. If $T \in \Sigma$ and $T \notin \mathcal{H}$, then $x \notin T$. Thus since Σ is a T_1-subbase for X there is some $S \in \Sigma$ such that $x \in S$ and $T \cap S = \emptyset$. It follows that $S \in \mathcal{H}$ and so $\mathcal{H} \cup \{T\}$ is not linked; therefore $\mathcal{H}$ is a maximal linked system of Σ.

Conversely, suppose now that Σ is a subbase for the closed subsets of a T_1-space X such that $\{S \in \Sigma: x \in S\}$ is a maximal linked system of Σ for each $x \in X$. Let x be a fixed but arbitrary point of X.

Clearly $\{x\} \subseteq \bigcap\{S \in \Sigma: x \in S\}$. If $y \in X$ and $y \neq x$, then it follows from the fact that X is a T_1-space that there is some $T \in \Sigma$ such that $x \in T$ and $y \notin T$. Therefore $y \notin \bigcap\{S \in \Sigma: x \in S\}$ and so $\bigcap\{S \in \Sigma: x \in S\} = \{x\}$. Finally, let $x \in X$ and $T \in \Sigma$ with $x \notin T$. Thus $T \notin \{S \in \Sigma: x \in S\}$ and since $\{S \in \Sigma: x \in S\}$ is a maximal linked system of Σ, there is some $S \in \Sigma$ such that $x \in S$ and $S \cap T = \emptyset$. Hence by Definition 2.11 (ii), Σ is a T_1-subbase for X.

NOTATION 2.14. Let Σ be a T_1-subbase for a space X.

$\lambda_\Sigma X$ will denote $\{\mathcal{L} \subseteq \Sigma: \mathcal{L}$ is a maximal linked system of $\Sigma\}$.

For each $S \in \Sigma$, S^+ will denote $\{\mathcal{L} \in \lambda_\Sigma X: S \in \mathcal{L}\}$, and Σ^+ will denote $\{S^+: S \in \Sigma\}$.

DEFINITION 2.15. Suppose that Σ is a T_1-subbase for a space X. The set $\lambda_\Sigma X$ equipped with the topology which has Σ^+ as a subbase for the closed sets is called the *superextension of X with respect to Σ*.

The following proposition states some results found in [7] which follow readily from the definitions. The proof is thus omitted.

PROPOSITION 2.16. *Suppose that Σ is a T_1-subbase for the space X.*

(i) *If $S, T \in \Sigma$ with $S \cap T = \emptyset$, then $S^+ \cap T^+ = \emptyset$.*

(ii) *If $\mathcal{L} \in \lambda_\Sigma X$ and $S \in \Sigma$ such that $\mathcal{L} \cup \{S\}$ is linked, then $S \in \mathcal{L}$.*

(iii) *If $\mathcal{L}$ is a linked system of Σ, then $\mathcal{L}$ is contained in some maximal linked system of Σ.*

(iv) *If $\mathcal{L} \in \lambda_\Sigma X$ and $\bigcap\mathcal{L} \neq \emptyset$, then $\bigcap\mathcal{L} = \{x\}$ for some $x \in X$.*

(v) If $e_\Sigma: X \to \lambda_\Sigma X$ is defined by $e_\Sigma(x) = \{S \in \Sigma: x \in S\}$, then e_Σ is an embedding of X into $\lambda_\Sigma X$, and $e_\Sigma[S] = S^+ \cap e_\Sigma[X]$ for each $S \in \Sigma$.

(vi) The space $\lambda_\Sigma X$ is T_1.

(vii) The space $\lambda_\Sigma X$ is compact.

(viii) If S and T are in Σ and $S \cup T = X$, then $S^+ \cup T^+ = \lambda_\Sigma X$.

DEFINITION 2.17. Let Σ be a T_1-subbase for a space X.

(i) Let $\beta_\Sigma X$ denote the topological closure of $e_\Sigma[X]$ in $\lambda_\Sigma X$; this will be called the de Groot compactification of X with respect to Σ. Note that $\{\beta_\Sigma X \cap S^+: S \in \Sigma\}$ is a subbase for the closed subsets of $\beta_\Sigma X$.

(ii) If $\mathcal{L} \subseteq \Sigma$, then $\mathcal{L}$ is said to be prime if and only if whenever $S_1, S_2, \cdots, S_n \in \Sigma$ such that $\cup\{S_i: i = 1,2,\cdots,n\} = X$, there exists some $j \in \{1,2,\cdots,n\}$ such that $S_j \in \mathcal{L}$.

(iii) A subset $\mathcal{L}$ of Σ is said to be centered if each non-void finite subcollection from $\mathcal{L}$ has a non-void intersection; $\mathcal{L}$ is called a maximal centered system of Σ in case $\mathcal{L}$ is maximal in Σ with respect to the property of being centered.

It should be clear from Definition 2.7 (iv) that $\beta_\Sigma X$ is indeed a compactification of X. However, $\beta_\Sigma X$ is not necessarily Hausdorff even if X is Hausdorff.

PROPOSITION 2.18. Suppose that Σ is a T_1-subbase for a space X.

(i) If $\mathcal{F}$ is a finite subcollection of Σ, then $\cup\mathcal{F} = X$ if and only if $\cup\{\beta_\Sigma X \cap T^+ : T \in \mathcal{F}\} = \beta_\Sigma X$.

(ii) If Σ is weakly-normal, then $\beta_\Sigma X$ is Hausdorff.

(iii) If $\mathcal{L} \in \lambda_\Sigma X$, then $\mathcal{L} \in \beta_\Sigma X$ if and only if $\mathcal{L}$ is prime.

(iv) If $\mathcal{L}$ is any maximal centered system of Σ, then $\mathcal{L}$ is prime.

(v) If Σ is weakly-normal and $\mathcal{L} \in \lambda_\Sigma X$, then $\mathcal{L} \in \beta_\Sigma X$ if and only if $\mathcal{L}$ contains a maximal centered system of Σ.

The proof of Proposition 2.18 can be found in [7] Chapter 1, Proposition 1.5 through Proposition 1.9.

Propositions 2.16 and 2.18 together with the fact that a T_1-space is completely regular if and only if it can be embedded in a compact Hausdorff space outline a proof for the following statement from [6]:

A T_1-space X is completely regular if and only if there exists a weakly-normal T_1-subbase for the closed subsets of X.

EXAMPLE 2.19. Suppose that X is a completely regular T_1-space and Σ is the collection of all non-void zero sets of X. Then Σ is a weakly-normal T_1-subbase for X, and $\beta_\Sigma X$ is the Stone-Čech compactification of X. (This follows from Proposition 2.16 (i) and Theorem 6.5 of [10].)

3. PRODUCTS OF SUPEREXTENSIONS

In this chapter we investigate the productivity of superextensions of T_1-spaces and show that a natural superextension of the product of an arbitrary family of T_1-spaces is the product of the superextensions of the family. We also show that the corresponding de Groot compactification of the product is the product of the de Groot compactifications. Throughout this chapter we let $\{Y_a\}_{a\in A}$ be a family of T_1-spaces, Σ_a be a T_1-subbase for the closed sets of Y_a, Y be the topological product of the family $\{Y_a\}_{a\in A}$, and for each $a \in A$, $\pi_a: Y \to Y_a$ be the projection function. Also without loss of generality we may assume that for each $a \in A$, the set Y_a is itself a member of the subbase Σ_a.

PROPOSITION 3.1. $\Sigma = \{\pi_a^{-1}[S]: S \in \Sigma_a, a \in A\}$ *is a* T_1*-subbase for the closed sets of* Y.

PROOF. It is clear that Σ is a subbase for the closed sets of the product space Y. Suppose that $y, z \in Y$ and $z \in \bigcap\{\pi_a^{-1}[T]: \pi_a^{-1}[T] \in \Sigma$ and $y \in \pi_a^{-1}[T]\}$. Thus for each $a \in A$, $\pi_a(z) \in \bigcap\{T: T \in \Sigma_a$ and $\pi_a(y) \in T\}$. Since Σ_a is a T_1-subbase for Y_a, $\bigcap\{T \in \Sigma_a: \pi_a(y) \in T\} = \{\pi_a(y)\}$; therefore $\pi_a(z) = \pi_a(y)$ for each $a \in A$ and hence $z = y$. Now let $y \in Y$ and $\pi_a^{-1}[S] \in \Sigma$ with $y \notin \pi_a^{-1}[S]$. If follows that $\pi_a(y) \notin S$, and

since Σ_a is a T_1-subbase for Y_a, there is some $T \in \Sigma_a$ such that $\pi_a(y) \in T$ and $S \cap T \neq \emptyset$. Therefore, $\pi_a^{-1}[T] \in \Sigma$, $y \in \pi_a^{-1}[T]$ and $\pi_a^{-1}[S] \cap \pi_a^{-1}[T] = \emptyset$. Hence Σ is a T_1-subbase for Y. (cf. Definition 2.11 (ii)).

In view of the above proposition we may consider the superextension of Y with respect to Σ.

PROPOSITION 3.2. Let P be the topological product of the family $\{\lambda_{\Sigma_a} Y_a\}_{a \in A}$. For each $(\mathcal{L}_a)_{a \in A} \in P$, the set $\mathcal{H} = \{Q \in \Sigma$: for each $a \in A$, $\pi_a[Q] \in \mathcal{L}_a\}$ is a maximal linked system of Σ.

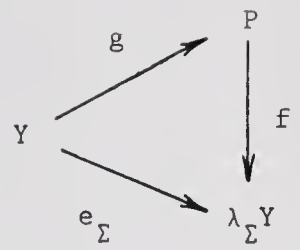
PROOF. Suppose that $Q, R \in \mathcal{H}$. For each $a \in A$, $\pi_a[Q], \pi_a[R] \in \mathcal{L}_a$ so that $\pi_a[Q] \cap \pi_a[R] \neq \emptyset$; therefore, $Q \cap R \neq \emptyset$ so that $\mathcal{H}$ is linked. Suppose now that $\pi_b^{-1}[S] \in \Sigma$ and $\mathcal{H} \cup \{\pi_b^{-1}[S]\}$ is linked. If $a \in A$ and $a \neq b$, then $\pi_a[\pi_b^{-1}[S]] = Y_a$. If $a = b$, then $\pi_a[\pi_b^{-1}[S]] = S$. Thus for each $a \in A$ it follows that $\pi_a[\pi_b^{-1}[S]] \in \mathcal{L}_a$ and so $\pi_b^{-1}[S] \in \mathcal{H}$; i.e., $\mathcal{H}$ is a maximal linked system of Σ.

THEOREM 3.3. There is a homeomorphism f from the product P of the superextensions $\{\lambda_{\Sigma_a} Y_a\}_{a \in A}$ to the superextension $\lambda_\Sigma Y$ of the product Y. Furthermore, if g is the unique continuous map which makes each of the diagrams

$$\begin{array}{ccc} Y & \xrightarrow{g} & P \\ \downarrow{\scriptstyle \pi_a} & & \downarrow{\scriptstyle \hat{\pi}_a} \\ Y_a & \xrightarrow[e_{\Sigma_a}]{} & \lambda_{\Sigma_a} Y_a \end{array}$$

commute,

then the diagram



commutes.

PROOF. By Proposition 3.2 there is a function $f: P \to \lambda_\Sigma Y$ defined by $f((\mathcal{L}_a)_{a\in A}) = \{Q \in \Sigma$: for each $a \in A$, $\pi_a[Q] \in \mathcal{L}_a\}$.

We will show that $f \circ g = e_\Sigma$. For each $y \in Y$, $f(g(y)) = \{Q \in \Sigma$: for each $a \in A$, $\pi_a[Q] \in \hat{\pi}_a(g(y))\} = \{Q \in \Sigma$: for each $a \in A$, $\pi_a[Q] \in e_{\Sigma_a}(\pi_a(y))\}$ $= \{Q \in \Sigma$: for each $a \in A$, $\pi_a(y) \in \pi_a[Q]\} = \{Q \in \Sigma: y \in Q\} = e_\Sigma(y)$.

To see that f is injective, let $\mathcal{J}$ and $\mathcal{K}$ be members of P such that $\mathcal{J} \neq \mathcal{K}$. Thus there must be some $b \in A$ such that $\hat{\pi}_b(\mathcal{J}) \neq \hat{\pi}_b(\mathcal{K})$, and since $\hat{\pi}_b(\mathcal{J})$ and $\hat{\pi}_b(\mathcal{K})$ are maximal linked systems of Σ_b, it follows that there exist $T \in \hat{\pi}_b(\mathcal{J})$ and $S \in \hat{\pi}_b(\mathcal{K})$ such that $S \cap T = \emptyset$. Hence $\pi_b^{-1}[S] \cap \pi_b^{-1}[T] = \emptyset$ and so $\pi_b^{-1}[S]$ and $\pi_b^{-1}[T]$ cannot be contained in the same maximal linked system of Σ. Recall that if $a \in A$ and $a \neq b$, then $\pi_a[\pi_b^{-1}[T]] = Y_a$, and if $a = b$, then $\pi_a[\pi_b^{-1}[T]] = T$. It follows that for each $a \in A$, $\pi_a[\pi_b^{-1}[T]] \in \hat{\pi}_a(\mathcal{J})$. Hence by the definition of f, $\pi_b^{-1}[T] \in f(\mathcal{J})$. By a similar argument we see that $\pi_b^{-1}[S] \in f(\mathcal{K})$. Thus $f(\mathcal{J}) \neq f(\mathcal{K})$.

To see that f is surjective, let $\mathcal{L} \in \lambda_\Sigma Y$. For each $a \in A$, let $\mathcal{L}_a = \{\pi_a[Q]: Q \in \mathcal{L}\}$. For each $a \in A$, $\mathcal{L}_a$ is clearly linked, and if $S \in \Sigma_a$ such that $\mathcal{L}_a \cup \{S\}$ is linked, then it follows that $\mathcal{L} \cup \{\pi_a^{-1}[S]\}$

is linked and hence $\pi_a^{-1}[S] \in \mathcal{L}$. Therefore $S = \pi_a[\pi_a^{-1}[S]] \in \mathcal{L}_a$ so that $\mathcal{L}_a$ is a maximal linked system of Σ_a. Thus $(\mathcal{L}_a)_{a\in A} \in P$ and by the definition of f, $f((\mathcal{L}_a)_{a\in A}) = \{Q \in \Sigma$: for each $a \in A$, $\pi_a[Q] \in \mathcal{L}_a\}$ $= \{Q \in \Sigma: Q \in \mathcal{L}\} = \mathcal{L}$.

The continuity of f is an immediate consequence of the fact that for each $b \in A$ and each $T \in \Sigma_b$, $f^{-1}[(\pi_b^{-1}[T])^+] = \{(\mathcal{L}_a)_{a\in A} \in P:$ $f((\mathcal{L}_a)_{a\in A}) \in (\pi_b^{-1}[T])^+\} = \{(\mathcal{L}_a)_{a\in A} \in P: \pi_b^{-1}[T] \in f((\mathcal{L}_a)_{a\in A})\}$ $= \{(\mathcal{L}_a)_{a\in A} \in P$: for each $a \in A$, $\pi_a[\pi_b^{-1}[T]] \in \mathcal{L}_a\} = \{(\mathcal{L}_a)_{a\in A} \in P:$ $T(= \pi_b[\pi_b^{-1}[T]]) \in \mathcal{L}_b\} = \hat{\pi}_b^{-1}[\{\mathcal{J} \in \lambda_{\Sigma_b} Y_b: T \in \mathcal{J}\}] = \hat{\pi}_b^{-1}[T^+]$ which is a closed subset of P.

Finally we show that f is a closed map. Using the fact that f is injective and applying f to the above equality we know that for each $a \in A$ and $T \in \Sigma_a$, $f[\hat{\pi}_a^{-1}[T^+]] = f[f^{-1}[(\pi_a^{-1}[T])^+]] = (\pi_a^{-1}[T])^+$. It follows from the fact that $\{\hat{\pi}_a^{-1}[T^+]: a \in A, T^+ \in \Sigma_a^+\}$ is a subbase for the closed sets of P and that f is bijective, that f is a closed map.

Consequently, $\lambda_\Sigma Y$ is homeomorphic with P.

Let Q be the topological product of the family $\{\beta_{\Sigma_a} Y_a: a \in A\}$. Thus Q is a subspace of the product space P.

COROLLARY 3.4. *There is a homeomorphism h from the product Q of the compactifications $\{\beta_{\Sigma_a} Y_a\}_{a\in A}$ to the de Groot compactification $\beta_\Sigma Y$ of the product Y.*

PROOF. Let h be the restriction of f in Theorem 3.3 to Q. Since f is a homeomorphism we need only show that $h[Q] = \beta_\Sigma Y$.

To see that $\beta_\Sigma Y \subseteq h[Q]$, let $\mathcal{L} \in \beta_\Sigma Y$. Recall that in the proof of Theorem 3.3 we showed that for each $a \in A$, $\mathcal{L}_a = \{\pi_a[S]: S \in \mathcal{L}\}$ is a maximal linked system of Σ_a and that $f((\mathcal{L}_a)_{a\in A}) = \mathcal{L}$. To see that $(\mathcal{L}_a)_{a\in A} \in Q$ we will show that for each $a \in A$, $\mathcal{L}_a$ is prime and Hence $\mathcal{L}_a \in \beta_{\Sigma_a} Y_a$ (cf., Proposition 2.18 (iii)). Let $a \in A$ and suppose that $\{T_1, T_2, \cdots, T_n\} \subseteq \Sigma_a$ such that $\cup\{T_i: i = 1, 2, \cdots, n\} = Y_a$. It follows that $\{\pi_a^{-1}[T_1], \pi_a^{-1}[T_2], \cdots, \pi_a^{-1}[T_n]\} \subseteq \Sigma$ and $\cup\{\pi_a^{-1}[T_i]: i = 1, 2, \cdots, n\} = Y$. Since $\mathcal{L}$ is in $\beta_\Sigma Y$ and hence is prime, there is some $j \in \{1, 2, \cdots, n\}$ such that $\pi_a^{-1}[T_j] \in \mathcal{L}$. Thus $\pi_a[\pi_a^{-1}[T_j]] = T_j \in \mathcal{L}_a$ and so $\mathcal{L}_a$ is prime.

To see that $h[Q] \subseteq \beta_\Sigma Y$, let $(\mathcal{L}_a)_{a\in A} \in Q$. Thus for each $a \in A$, $\mathcal{L}_a \in \beta_{\Sigma_a} Y_a$ and so $\mathcal{L}_a$ is prime. Suppose now that Ψ is a finite subset of Σ such that $\cup\Psi = Y$. Recall that each member of Ψ is of the form $\pi_a^{-1}[T]$ for some $a \in A$ and $T \in \Sigma_a$. Let $B = \{a \in A:$ there is some $T \in \Sigma_a$ with $\pi_a^{-1}[T] \in \Psi\}$. Suppose that for each $b \in B$ there is some $y_b \in Y_b - \cup\{T \in \Sigma_b: \pi_b^{-1}[T] \in \Psi\}$. Then we can pick $y \in Y$ such that $\pi_b(y) = y_b$ for each $b \in B$. It follows that for each $\pi_a^{-1}[T] \in \Psi$, $y \notin \pi_a^{-1}[T]$ and so $y \in Y - \cup\Psi$--a contradiction. Hence there must be some $b \in B$ such that $\cup\{T \in \Sigma_b: \pi_b^{-1}[T] \in \Psi\} = Y_b$. Since $\mathcal{L}_b$ is prime, there is some $S \in \Sigma_b$ such that $\pi_b^{-1}[S] \in \Psi$ and $S \in \mathcal{L}_b$. It follows from the definition of f that $\pi_b^{-1}[S] \in f((\mathcal{L}_a)_{a\in A})$. Thus $f((\mathcal{L}_a)_{a\in A})$ is prime and so $h((\mathcal{L}_a)_{a\in A}) = f((\mathcal{L}_a)_{a\in A}) \in \beta_\Sigma X$. Consequently, h is a homeomorphism from Q onto $\beta_\Sigma Y$.

4. LATTICE SUPEREXTENSIONS

For the remainder of the paper we restrict our attention to superextensions of topological lattices with respect to T_1-subbases which consist of ideals and dual ideals. For these "admissible" subbases we show that the superextensions are actually de Groot compactifications (Theorem 4.6). Moreover, (with respect to a "naturally induced" ordering), they are shown to be complete lattice extensions of the original lattice (Theorem 4.11).

NOTATION 4.1. (i) For each $a \in X$, $X_D(a) = \{x \in X: x \leq a\}$ and $X_I(a) = \{x \in X: a \leq x\}$. (D denotes decreasing and I denotes increasing.)

(ii) $\Gamma_D = \{C \subseteq X: C \text{ is a closed ideal of } X\}$, $\Gamma_I = \{C \subseteq X: C \text{ is a closed dual ideal of } X\}$, and $\Gamma = \Gamma_D \cup \Gamma_I$.

(iii) $T = \{X\} \cup \{X_D(a): a \in X\} \cup \{X_I(a): a \in X\}$. (We refer to T as the usual subbase for X.)

(iv) If Σ is a subbase for the closed sets of X and $\Sigma \subseteq \Gamma$, then $\Sigma_I = \Sigma \cap \Gamma_I$ and $\Sigma_D = \Sigma \cap \Gamma_D$.

DEFINITION 4.2. (i) If Σ is a T_1-subbase for the closed subsets of X and $\Sigma \subseteq \Gamma$, then Σ is called an admissible subbase for X.

(ii) A lattice superextension of X is any superextension of the form $\lambda_\Sigma X$ where Σ is an admissible subbase for X.

PROPOSITION 4.3. If $T \subseteq \Sigma \subseteq \Gamma$, then Σ is an admissible subbase for X.

PROOF. It is clear from the definition of the interval topology (2.8 (ii)) that T is a subbase for the closed subsets of X. It is also clear that any collection of closed subsets of X containing T is a subbase for the closed sets of X. Hence Σ is a subbase for X. To see that Σ is a T_1-subbase for X, let $a \in X$. Thus since $T \subseteq \Sigma$, $\{a\} \subseteq \bigcap\{S \in \Sigma: a \in S\} \subseteq \bigcap\{U \in T: a \in U\} \subseteq X_D(a) \cap X_I(a) = \{a\}$; i.e., $\bigcap\{S \in \Sigma: a \in S\} = \{a\}$. Finally, suppose $a \in X$ and $S \in \Sigma$ such that $a \notin S$. If $S \in \Sigma_D$, then $X_I(a) \in \Sigma_I$, $a \in X_I(a)$ and $X_I(a) \cap S = \emptyset$. Dually, if $S \in \Sigma_I$, then $X_D(a) \in \Sigma_D$, $a \in X_D(a)$ and $X_D(a) \cap S = \emptyset$. Thus Σ is a T_1-subbase for X (cf., Definition 2.11 (ii)) and hence is an admissible subbase for X (4.2 (i)).

COROLLARY 4.4. T and Γ are admissible subbases for X.

By the above corollary $\lambda_T X$ and $\lambda_\Gamma X$ are lattice superextensions of X. Examples of these can be found in Chapter 8. Next we establish that all lattice superextensions of X are in fact compactifications of X which can be considered as complete lattices having X as a sublattice.

PROPOSITION 4.5. Let Σ be an admissible subbase for X.

(i) Σ_D and Σ_I are centered systems of Σ.

(ii) Every linked system of Σ is centered.

PROOF. (i) is immediate from Proposition 2.4 (ii).

(ii) Suppose that $\mathcal{L}$ is a linked system of Σ and $F_1, F_2, \cdots, F_n \in \mathcal{L}$. If all of the F_i belong to Σ_D, then by (i) above $\cap\{F_i: i = 1, 2, \cdots, n\} \neq \emptyset$. Similarly, if all of the F_i belong to Σ_I, then $\cap\{F_i: i = 1, 2, \cdots, n\} \neq \emptyset$. In the remaining case we may assume, by at most rearranging the subscripts, that there is some integer m such that $1 \leq m \leq n - 1$, $F_i \in \Sigma_D$ for $i = 1, \cdots, m$, and $F_i \in \Sigma_I$ for $i = m + 1, \cdots, n$. For each $i \in \{1, \cdots, m\}$ and $j \in \{m + 1, \cdots, n\}$ we can pick $f_{i,j} \in F_i \cap F_j$ because $\mathcal{L}$ is linked. Since the members of Σ_D are decreasing and those of Σ_I are dual ideals, it follows that for each $j \in \{m + 1, \cdots, n\}$, $g_j = \wedge\{f_{i,j}: i = 1, \cdots, m\} \in F_j \cap (\cap\{F_i: i = 1, \cdots, m\})$. In the same way, since the members of Σ_I are increasing and $\cap\{F_i: i = 1, \cdots, m\}$ is an ideal (2.4 (ii)), it follows that $\vee\{g_j: j = m + 1, \cdots, n\} \in (\cap\{F_i: i = 1, \cdots, m\}) \cap (\cap\{F_j: j = m + 1, \cdots, n\})$; i.e., $\mathcal{L}$ is centered.

THEOREM 4.6. *Each superextension of X with respect to an admissible subbase Σ is a de Groot compactification of X; i.e., $\lambda_\Sigma X = \beta_\Sigma X$.*

PROOF. Suppose that Σ is an admissible subbase for X. It is clear from the definitions in Chapter 2 that $\beta_\Sigma X \subseteq \lambda_\Sigma X$; therefore, we need only show that $\lambda_\Sigma X \subseteq \beta_\Sigma X$. If $\mathcal{L}$ is a maximal linked system of Σ, then by Proposition 4.5 (ii), $\mathcal{L}$ is centered; i.e., $\mathcal{L}$ is a maximal centered system of Σ. Thus $\mathcal{L}$ is prime (2.18 (iv)), and so $\mathcal{L} \in \beta_\Sigma X$ (2.18 (iii)).

In view of Theorem 4.6 every lattice superextension of X is actually a topological compactification of X. To see that it is also a complete lattice with the interval topology and that it has X as a sublattice we continue as follows.

DEFINITION 4.7. Suppose that Σ is an admissible subbase for X and $\mathcal{L}, \mathcal{M} \in \lambda_\Sigma X$. We say that $\mathcal{L} \leq \mathcal{M}$ whenever $\mathcal{M} \cap \Sigma_D \subseteq \mathcal{L} \cap \Sigma_D$, or, dually, whenever $\mathcal{L} \cap \Sigma_I \subseteq \mathcal{M} \cap \Sigma_I$.

Since containment is a partial ordering, it follows readily that 4.7 defines a partial ordering on the elements of $\lambda_\Sigma X$. Throughout the remainder of this paper, $\lambda_\Sigma X$ will be understood to be partially ordered according to Definition 4.7 whenever Σ is an admissible subbase for X. The following theorem shows that $\lambda_\Sigma X$ is a complete lattice whenever Σ is admissible.

THEOREM 4.8. *Suppose that Σ is an admissible subbase for X. Every non-void subset of $\lambda_\Sigma X$ has a supremum in $\lambda_\Sigma X$ and an infimum in $\lambda_\Sigma X$. Specifically, if $\emptyset \neq C \subseteq \lambda_\Sigma X$ and if*

$$C_D = \cap\{\mathcal{L} \cap \Sigma_D : \mathcal{L} \in C\} \quad \text{and}$$
$$C_I = \cap\{\mathcal{L} \cap \Sigma_I : \mathcal{L} \in C\},$$

then

$$\vee C = C_D \cup \{S \in \Sigma_I : C_D \cup \{S\} \text{ is linked}\} \text{ and}$$
$$\wedge C = C_I \cup \{S \in \Sigma_D : C_I \cup \{S\} \text{ is linked}\}.$$

PROOF. It is easily seen that $\mathcal{J} = C_D \cup \{S \in \Sigma_I : C_D \cup \{S\}$ is linked$\}$ is a linked system of Σ. Suppose that $T \in \Sigma_D$ and $\mathcal{J} \cup \{T\}$ is linked. Since $\mathcal{L} \cap \Sigma_I \subseteq \mathcal{J} \cap \Sigma_I$ for each $\mathcal{L} \in C$, it follows that $\mathcal{L} \cup \{T\}$ is linked for each $\mathcal{L} \in C$. Thus $T \in \mathcal{L}$ for each $\mathcal{L} \in C$ and so $T \in C_D$; hence $T \in \mathcal{J}$. If $T \in \Sigma_I$ and $\mathcal{J} \cup \{T\}$ is linked, then $C_D \cup \{T\}$ is linked and so by definition, $T \in \mathcal{J}$. Therefore $\mathcal{J}$ is a maximal linked system of Σ. A dual argument shows that $\mathcal{K} = C_I \cup \{S \in \Sigma_D : C_I \cup \{S\}$ is linked$\}$ is a maximal linked system of Σ.

We continue by showing that $\mathcal{J} = \vee C$. Since $\mathcal{L} \cap \Sigma_I \subseteq \mathcal{J} \cap \Sigma_I$ for each $\mathcal{L} \in C$, then by Definition 4.7, $\mathcal{L} \leq \mathcal{J}$ for each $\mathcal{L} \in C$. If $\mathcal{M} \in \lambda_\Sigma X$ and $\mathcal{M} < \mathcal{J}$, then there is some T in $(\mathcal{M} \cap \Sigma_D) - (\mathcal{J} \cap \Sigma_D)$. Since $T \notin \mathcal{J} \cap \Sigma_D = C_D$, there is some $\mathcal{L} \in C$ such that $T \notin \mathcal{L} \cap \Sigma_D$. It follows from this that $\mathcal{L} \not\leq \mathcal{M}$; therefore, $\mathcal{J} = \vee C$. A dual argument shows that $\mathcal{K} = \wedge C$.

PROPOSITION 4.9. *Let Σ be an admissible subbase for X. Then the natural embedding $e_\Sigma : X \to \lambda_\Sigma X$ defined by $x \mapsto \{S \in \Sigma : x \in S\}$ is a lattice inclusion; thus X is a sublattice of $\lambda_\Sigma X$.*

PROOF. That e_Σ is injective follows from the fact that it is a topological embedding (Proposition 2.16 (v)). To see that it

preserves suprema, suppose that $C \subseteq X$ such that $\vee_X C$ exists in X. If $S \in e_\Sigma(\vee_X C) \cap \Sigma_D$, then $\vee_X C \in S$, and since S is decreasing it follows that $C \subseteq S$. Therefore $S \in e_\Sigma(c) \cap \Sigma_D$ for each $c \in C$ and by Proposition 4.8, $S \in \vee e_\Sigma[C]$. Hence by Definition 4.7, $\vee e_\Sigma[C] \leq e_\Sigma(\vee_X C)$. Conversely, if $S \in \vee e_\Sigma[C] \cap \Sigma_D$, then $S \in e_\Sigma(c) \cap \Sigma_D$ and $c \in S$ for each $c \in C$. Thus $C \subseteq S$ and, since S is a closed ideal of X, it follows from Frink [9; Theorem 12] that $\vee_X C \in S$. Hence $S \in e_\Sigma(\vee_X C)$, and by Definition 4.7, $e_\Sigma(\vee_X C) \leq \vee e_\Sigma[C]$; i.e., $e_\Sigma(\vee_X C) = \vee e_\Sigma[C]$. A dual argument shows that if $C \subseteq X$ and $\wedge_X C$ exists in X, then $e_\Sigma(\wedge_X C) = \wedge e_\Sigma[C]$. By Definition 2.2 (ii), e_Σ is a lattice inclusion.

PROPOSITION 4.10. For any admissible subbase Σ of X, the following hold: (Recall that for each $S \in \Sigma$, $S^+ = \{\mathcal{L} \in \lambda_\Sigma X: S \in \mathcal{L}\}$.)

(i) If $S \in \Sigma_D$ and $T \in \Sigma_I$, then $S^+ = \{\mathcal{L} \in \lambda_\Sigma X: \mathcal{L} \leq \vee S^+\}$ and $T^+ = \{\mathcal{L} \in \lambda_\Sigma X: \wedge T^+ \leq \mathcal{L}\}$.

(ii) For each $\mathcal{M} \in \lambda_\Sigma X$, $\{\mathcal{L} \in \lambda_\Sigma X: \mathcal{L} \leq \mathcal{M}\} = \cap\{S^+: S \in \mathcal{M} \cap \Sigma_D\}$ and $\{\mathcal{L} \in \lambda_\Sigma X: \mathcal{M} \leq \mathcal{L}\} = \cap\{S^+: S \in \mathcal{M} \cap \Sigma_I\}$.

(iii) For each $\mathcal{M} \in \lambda_\Sigma X$, $e_\Sigma^{-1}[\{\mathcal{L} \in \lambda_\Sigma X: \mathcal{L} \leq \mathcal{M}\}] = \cap(\mathcal{M} \cap \Sigma_D)$ and $e_\Sigma^{-1}[\{\mathcal{L} \in \lambda_\Sigma X: \mathcal{M} \leq \mathcal{L}\}] = \cap(\mathcal{M} \cap \Sigma_I)$.

(iv) For each $T \in \Sigma_D$, $\vee T^+ = \vee e_\Sigma[T]$, and for each $T \in \Sigma_I$, $\wedge T^+ = \wedge e_\Sigma[T]$.

(v) If $T \in \Sigma_D$ and $S \in \Sigma_I$, then $S \cap T \neq \emptyset$ if and only if $\wedge S^+ \leq \vee T^+$.

(vi) If $\mathcal{L} \in \lambda_\Sigma X$, $T \in \Sigma_D$, and $S \in \Sigma_I$, then $T \in \mathcal{L}$ if and only if $\mathcal{L} \leq \vee T^+$, and $S \in \mathcal{L}$ if and only if $\wedge S^+ \leq \mathcal{L}$.

(vii) For each $\mathcal{L} \in \lambda_\Sigma X$, $\vee\{\wedge S^+: S \in \mathcal{L} \cap \Sigma_I\} = \mathcal{L} = \wedge\{\vee S^+: S \in \mathcal{L} \cap \Sigma_D\}$.

PROOF. (i) Note first that by Proposition 4.8, $\vee S^+ \in \lambda_\Sigma X$ and $S \in \vee S^+$. Clearly, if $\mathcal{L} \in S^+$, then $\mathcal{L} \leq \vee S^+$. Conversely, if $\mathcal{L} \leq \vee S^+$, then $S \in \vee S^+ \cap \Sigma_D \subseteq \mathcal{L} \cap \Sigma_D$ and so $\mathcal{L} \in S^+$. The second equality is proved dually.

(ii) Suppose that $\mathcal{M} \in \lambda_\Sigma X$. It follows from Definition 4.7 that $\{\mathcal{L} \in \lambda_\Sigma X: \mathcal{L} \leq \mathcal{M}\} \subseteq \cap\{S^+: S \in \mathcal{M} \cap \Sigma_D\}$. Conversely, if $\mathcal{L} \in \lambda_\Sigma X$ and $\mathcal{L} \not\leq \mathcal{M}$, then again by Definition 4.7 there is some $T \in (\mathcal{M} \cap \Sigma_D) - (\mathcal{L} \cap \Sigma_D)$. Thus $\mathcal{L} \notin T^+$ and so $\mathcal{L} \notin \cap\{S^+: S \in \mathcal{M} \cap \Sigma_D\}$. The second equality is proved dually.

(iii) Recall from Proposition 2.16 (v) that for each $T \in \Sigma$, $T^+ \cap e_\Sigma[X] = e_\Sigma[T]$. Thus since e_Σ is injective, it follows that for each $T \in \Sigma$, $T = e_\Sigma^{-1}[e_\Sigma[T]] = e_\Sigma^{-1}[T^+ \cap e_\Sigma[X]] = e_\Sigma^{-1}[T^+]$. Let $\mathcal{M} \in \lambda_\Sigma X$. By the above remark and (ii) we see that $e_\Sigma^{-1}[\{\mathcal{L} \in \lambda_\Sigma X: \mathcal{L} \leq \mathcal{M}\}]$ $= e_\Sigma^{-1}[\cap\{S^+: S \in \mathcal{M} \cap \Sigma_D\}] = \cap\{e_\Sigma^{-1}[S^+]: S \in \mathcal{M} \cap \Sigma_D\}$ $= \cap\{S: S \in \mathcal{M} \cap \Sigma_D\} = \cap(\mathcal{M} \cap \Sigma_D)$. The other equality is proved similarly.

(iv) Let $T \in \Sigma_D$. Again using Proposition 2.16 (v) we see that $e_\Sigma[T] \subseteq T^+$ and so $\vee e_\Sigma[T] \leq \vee T^+$. If $S \in \vee e_\Sigma[T] \cap \Sigma_D$, then it follows from Proposition 4.8 that $T \subseteq S$ and so $S \in \vee T^+$; therefore by Definition 4.7, $\vee T^+ \leq \vee e_\Sigma[T]$ and hence $\vee T^+ = \vee e_\Sigma[T]$. A dual argument shows that if $T \in \Sigma_I$, then $\wedge T^+ = \wedge e_\Sigma[T]$.

(v) If there is some $x \in T \cap S$, then by (iv) above $\wedge S^+ \leq e_\Sigma(x) \leq \vee T^+$. Conversely, suppose that $\mathcal{L} = \wedge S^+ \leq \vee T^+$. By (i) above $\mathcal{L} \in S^+$ and $\mathcal{L} \in T^+$. Thus $S^+ \cap T^+ \neq \emptyset$ and hence $S \cap T \neq \emptyset$ (2.16 (i)).

(vi) Suppose that $\mathcal{L} \in \lambda_\Sigma X$ and $T \in \Sigma_D$. By (i) above $\mathcal{L} \leq \vee T^+$ if and only if $\mathcal{L} \in T^+$, and by definition, $\mathcal{L} \in T^+$ if and only if $T \in \mathcal{L}$. Thus $T \in \mathcal{L}$ if and only if $\mathcal{L} \leq \vee T^+$. Dually, if $S \in \Sigma_I$, then $\wedge S^+ \leq \mathcal{L}$ if and only if $S \in \mathcal{L}$.

(vii) Suppose that $\mathcal{L} \in \lambda_\Sigma X$. Let $\mathcal{J} = \vee \{\wedge S^+ : S \in \mathcal{L} \cap \Sigma_I\}$, and $\mathcal{K} = \wedge \{\vee T^+ : T \in \mathcal{L} \cap \Sigma_D\}$. By (vi) above, for each $S \in \mathcal{L} \cap \Sigma_I$ and each $T \in \mathcal{L} \cap \Sigma_D$, $\wedge S^+ \leq \mathcal{L} \leq \vee T^+$ so that $\mathcal{J} \leq \mathcal{L} \leq \mathcal{K}$. Let $R \in \mathcal{J} \cap \Sigma_D$. Thus by (vi), $\mathcal{J} \leq \vee R^+$. Since for each $S \in \mathcal{L} \cap \Sigma_I$ $\wedge S^+ \leq \mathcal{J}$, it follows from (v) that $S \cap R \neq \emptyset$ for each $S \in \mathcal{L} \cap \Sigma_I$; i.e., $(\mathcal{L} \cap \Sigma_I) \cup \{R\}$ is linked. Therefore $\mathcal{L} \cup \{R\}$ is linked, and so $R \in \mathcal{L}$. Hence by (v) and the definition of $\mathcal{K}$, $R \in \mathcal{K}$. By Definition 4.7, $\mathcal{K} \leq \mathcal{J}$ and hence $\mathcal{J} = \mathcal{K} = \mathcal{L}$.

THEOREM 4.11. Any lattice superextension of X is a complete topological lattice having X as a sublattice.

PROOF. By Theorem 4.8 and Proposition 4.9 any lattice superextension of X is a complete lattice having X as a sublattice. By Proposition 4.10 (i), the given subbase for a lattice superextension $\lambda_\Sigma X$ is a collection of principal ideals and principal dual ideals of $\lambda_\Sigma X$, and by Proposition 4.10 (ii) each principal ideal and principal dual ideal of $\lambda_\Sigma X$ is closed in the given topology on $\lambda_\Sigma X$. Thus the topology on $\lambda_\Sigma X$ generated by Σ^+ is the interval topology on $\lambda_\Sigma X$; i.e., $\lambda_\Sigma X$ is a topological lattice.

We can conclude this chapter by showing that in the case where X is totally ordered, all of the lattice superextensions of X are also totally ordered.

PROPOSITION 4.12. *If X is totally ordered and Σ is an admissible subbase for X, then Σ is weakly-normal and $\lambda_\Sigma X$ is totally ordered.*

PROOF. To see that Σ is weakly-normal, let $S, T \in \Sigma$ such that $S \cap T = \emptyset$. Either $S \cup T = X$, in which case $\{S,T\}$ is a screen for S and T, or there is some $x \in X - (S \cup T)$. In the latter case, since Σ is a T_1-subbase for X we may pick $S', T' \in \Sigma$ such that $x \in S' \cap T'$, $S \cap S' = \emptyset$, and $T \cap T' = \emptyset$. It now follows from the facts that X is totally ordered and that S and T are not both in Σ_D or Σ_I that $X = X_D(x) \cup X_I(x) \subseteq S' \cup T'$. Thus $\{S',T'\}$ is a screen for S and T. Hence by Definition 2.11 (iii), Σ is weakly-normal. To show that $\lambda_\Sigma X$ is totally ordered, suppose that $\mathcal{G}$ and $\mathcal{H}$ are members of $\lambda_\Sigma X$ with $\mathcal{H} \neq \mathcal{G}$. Then there exist $T \in \mathcal{G} \cap \Sigma_D$ and $S \in \mathcal{H} \cap \Sigma_I$ such that $S \cap T = \emptyset$. Since X is totally ordered it must be the case that $t \leq s$ for each $t \in T$ and $s \in S$. If $R \in \mathcal{H} \cap \Sigma_D$, then it follows from $R \cap S \neq \emptyset$ that $T \subseteq R$, $R \in \mathcal{G} \cap \Sigma_D$, and thus $\mathcal{G} \leq \mathcal{H}$.

5. THE COMPLETION BY CUTS

In this chapter we characterize the completion by cuts of a topological lattice as a particular lattice superextension (Theorem 5.13). In doing so we prove a theorem (5.11) which is used extensively in proving the main results in this and the succeeding chapter.

Throughout this chapter we let X denote a topological lattice. We first characterize the completion of X by cuts as a particular complete sublattice of the power set, 2^X, of X which is ordered by inclusion. Then using the theorem mentioned above we show that the completion of X by cuts is iseomorphic with the superextension of X with respect to the usual subbase T. We conclude the chapter by considering the case where X is totally ordered and investigating the conditions on X under which the completion by cuts is homeomorphic with the Alexandroff one-point compactification.

DEFINITION 5.1. A completion-operator on a lattice L is a map $\Phi: 2^L \to 2^L$ which satisfies the following conditions:

C1. $A \subseteq \Phi(A)$, for each $A \in 2^L$.

C2. $\Phi(A) = \Phi(\Phi(A))$, for each $A \in 2^L$.

C3. If A and B are in 2^L and $A \subseteq B$, then $\Phi(A) \subseteq \Phi(B)$.

REMARK 5.2. It follows from Ward [15] that if Φ is a completion-operator on L, then $\Delta = \{A \in 2^L: A = \Phi(A)\}$ is a complete lattice, ordered by inclusion. Moreover, for each $\Psi \subseteq \Delta$,

(i) $\wedge \Psi = \cap \Psi$ and

(ii) $\vee \Psi = \Phi(\cup \Psi)$,

and the mapping from L to Δ defined by $x \mapsto \Phi(\{x\})$ is an injective, order-preserving map.

Next, we consider a specific completion-operator on X.

PROPOSITION 5.3. The map $D: 2^X \to 2^X$ defined by $A \mapsto \cap\{X_D(a): A \subseteq X_D(a)\}$ is a completion-operator on X.

PROOF. C1: Clearly, for each $A \in 2^X$, $A \subseteq \cap\{X_D(a): A \subseteq X_D(a)\}$.

C2: For each $A \in 2^X$, by C1, $D(A) \subseteq D(D(A))$. Conversely, if $A \subseteq X_D(a)$ for some $a \in X$, then by definition, $D(A) \subseteq X_D(a)$; therefore, $D(D(A)) \subseteq D(A)$.

C3: Suppose that $A \subseteq B \subseteq X$. If $D(B) = X$, then clearly $D(A) \subseteq D(B)$. If, however, $B \subseteq X_D(a)$ for some $a \in X$, then, since $A \subseteq B \subseteq X_D(a)$, it follows that $D(A) \subseteq D(B)$.

It should be noted that for each $a \in X$, $D(\{a\}) = X_D(a)$ and that D(A) is the set of all predecessors of all successors of all members of A; i.e., $D(A) = \{x \in X: x \leq a$ for all $a \in X$ with the property that $b \leq a$ for all $b \in A\}$.

It follows from Proposition 5.3 and Remark 5.2 that $\tilde{X} = \{A \subseteq X: D(A) = A\}$ (when partially ordered by inclusion) is a complete lattice and that the map $d: X \to \tilde{X}$ defined by $a \mapsto D(\{a\})$ is injective and order-preserving. In view of the equivalence for D(A) noted in the preceding paragraph and Birkhoff [5 ; Chapter V, §9], $\tilde{X}$ is the completion of X by cuts.

PROPOSITION 5.4. *The injection* $d: X \to \tilde{X}$ *defined by* $x \mapsto D(\{x\})$ *is a lattice inclusion.*

PROOF. Since d is injective, we need only show that d is a lattice homomorphism (2.2 (i)). Suppose that C is a non-void subset of X such that $\vee C$ exists in X. It follows from Remark 5.2 that $\vee_{\tilde{X}} d[C] = D(\cup\{d(x): x \in C\}) = D(C)$. Let $c = \vee_X C$. Thus $C \subseteq X_D(c)$ and hence $D(C) \leq D(\{c\})$; i.e., $\vee_{\tilde{X}} d[C] \leq d(\vee_X C)$. Finally, if $a \in X$ such that $C \subseteq X_D(a)$, then a is an upper bound of C; hence $c = \vee_X C \leq a$ so that $c \in X_D(a)$. It follows that $d(c) = d(\vee_X C) \leq D(C) = \vee_{\tilde{X}} d[C]$. Thus $d(\vee_X C) = \vee_{\tilde{X}} d[C]$. A dual argument shows that if $\emptyset \neq C \subseteq X$ such that $\wedge_X C$ exists, then $d(\wedge_X C) = \wedge_{\tilde{X}} d[C]$. Thus d is a lattice homomorphism (2.2 (i)).

Next, we demonstrate that $\tilde{X}$ (when it is equipped with its interval topology) is also a compactification of X.

PROPOSITION 5.5. For each $A \in \tilde{X}$

(i) $\wedge \{d(a): A \leq d(a)\} = A = \vee \{d(a): a \in A\}$,

(ii) $\tilde{X}_D(A) = \cap\{\tilde{X}_D(d(a)): A \leq d(a)\}$,

(iii) $\tilde{X}_I(A) = \cap\{\tilde{X}_I(d(a)): a \in A\}$.

PROOF. Let $A \in \tilde{X}$.

(i) Since $a \in A$ if and only if $d(a) \leq A$, it is clear that

$$\vee \{d(a): a \in A\} \leq A \leq \wedge \{d(a): A \leq d(a)\}.$$

To see that $\vee \{d(a): a \in A\} = A$, let $B \in \tilde{X}$ such that $B < A$. Since $\tilde{X}$ is ordered by inclusion there is some $a \in A$ such that $a \notin B$; therefore $X_D(a) \not\subseteq B$ so that $d(a) \not\leq B$. Thus $\vee \{d(a): a \in A\} = A$.

To see that $\wedge \{d(a): A \leq d(a)\} = A$, suppose that $B \in X$ such that $A < B$. Since $D(A) \neq D(B)$, it follows from the definition of D that there is some $a \in X$ such that $A \subseteq X_D(a)$ and $B \not\subseteq X_D(a)$. Thus $A \leq d(a)$ and $B \not\leq d(a)$. It follows that $A = \wedge \{d(a): A \leq d(a)\}$.

(ii) Since $A \leq d(a)$ implies that $\tilde{X}_D(A) \subseteq \tilde{X}_D(d(a))$, it is clear that

$$\tilde{X}_D(A) \subseteq \cap\{\tilde{X}_D(d(a)): A \leq d(a)\}.$$

To see the other inclusion, suppose that $B \in \tilde{X}$ such that $B \in \cap\{\tilde{X}_D(d(a)): A \leq d(a)\}$. Thus $B \leq d(a)$ for each $a \in X$ such that $A \leq d(a)$. By (i), $A = \wedge \{d(a): A \leq d(a)\}$ and it follows that $B \leq A$; i.e., $B \in \tilde{X}_D(A)$.

(iii) Since $a \in A$ if and only if $d(a) \leq A$, (iii) can be proved by dualizing the argument in (ii).

PROPOSITION 5.6. For each $a \in X$, $\overline{d[X_D(a)]} = \tilde{X}_D(d(a))$ and $\overline{d[X_I(a)]} = \tilde{X}_I(d(a))$.

PROOF. Let $a \in X$. Since d preserves the order of X, it is clear that $d[X_D(a)] \subseteq \tilde{X}_D(d(a))$, and since $\tilde{X}_D(d(a))$ is closed, $\overline{d[X_D(a)]} \subseteq \tilde{X}_D(d(a))$. To see the other inclusion, let $A \in \tilde{X}_D(d(a))$ and let W be a basic open set containing A.

If W is of the form $\tilde{X} - \cup\{\tilde{X}_I(B_i): i = 1, \cdots, n\}$ for some finite set $\{B_1, \cdots, B_n\} \subseteq \tilde{X}$, then since $A \in W$, it must be the case that for each i, $\wedge \tilde{X} < B_i$. Since, by Proposition 5.5 (i), $\wedge \tilde{X} = \wedge \{d(x): \wedge \tilde{X} \leq d(x)\}$, it follows that for each $i \in \{1, \cdots, n\}$ there is some $b_i \in X$ such that $\wedge \tilde{X} \leq d(b_i)$ and $B_i \not\leq d(b_i)$. Let $c = (\wedge \{b_i: i = 1, \cdots, n\}) \wedge a$. Then $c \in X$, $c \leq a$, and since d is a lattice inclusion, $d(c) \leq d(a)$. Thus $d(c) \in d[X_D(a)]$. Finally, note that $d(c) \in W$, for if not, then $d(c) \in \tilde{X}_I(B_i)$ for some $i \in \{1, \cdots, n\}$, and so $B_i \leq d(c) \leq d(b_i)$ —a contradiction. Hence $W \cap d[X_D(a)] \neq \emptyset$.

Consider now the case that W has the form $\tilde{X} - \cup\{\tilde{X}_D(C_j): j = 1, \cdots, m$ for some finite set $\{C_1, \cdots, C_m\} \subseteq \tilde{X}$. Since for each $j = 1, \cdots, m$, $A \not\leq C_j$, and since $A \leq d(a)$, it follows that for each j, $d(a) \not\leq C_j$. Thus $d(a) \in W \cap d[X_D(a)]$. Hence $W \cap d[X_D(a)] \neq \emptyset$.

Finally, we consider the case that $W = V \cap U$ where $V = \tilde{X} - \cup\{\tilde{X}_D(C_j): j = 1, \cdots, m\}$ and $U = \tilde{X} - \cup\{\tilde{X}_I(B_i): i = 1, \cdots, n\}$ for some finite set $\{B_1, \cdots, B_n, C_1, \cdots, C_m\} \subseteq \tilde{X}$. Since $A \in V$, it must be the case that $A \neq \wedge X$ and so $A \neq \emptyset$. Since $A \not\leq C_j$ and $A = \vee \{d(x): x \in A\}$ (5.5 (i)), it follows that for each $j \in \{1, \cdots, m\}$ there is some

$x_j \in A$ such that $d(x_j) \not\leq C_j$. Let $x = \vee \{x_j : j = 1, \cdots, m\}$. Since d is a lattice inclusion, it follows that for each j, $d(x_j) \leq d(x)$. Also $d(x) \leq A$, and since $A \leq d(a)$, $d(x) \in d[X_D(a)]$. Note that $d(x) \in V$, for if not, then there is some $j \in \{1, \cdots, m\}$ with $d(x) \in \tilde{X}_D(C_j)$ and thus $d(x_j) \leq d(x) \leq C_j$ -- a contradiction. Note also that $d(x) \in U$, for if not, then there is some $i \in \{1, \cdots, n\}$ with $d(x) \in \tilde{X}_I(B_i)$; thus $B_i \leq d(x) \leq A$ so that $A \notin U$ -- again a contradiction. Consequently, $d(x) \in W \cap d[X_D(a)]$. Hence, in each of the possible cases, $W \cap d[X_D(a)] \neq \emptyset$, so that, since W was arbitrarily chosen, $A \in \overline{d[X_D(a)]}$. A dual argument shows that $\overline{d[X_I(a)]} = \tilde{X}_I(d(a))$.

LEMMA 5.7. The collection $\tilde{T} = \{\tilde{X}_D(d(a)) : a \in X\} \cup \{\tilde{X}_I(d(a)) : a \in X\} \cup \{\tilde{X}\}$ is a subbase for $\tilde{X}$, $\{d^{-1}[A] : A \in \tilde{T}_D\} = T_D$, and $\{d^{-1}[A] : A \in \tilde{T}_I\} = T_I$.

PROOF. Clearly $\tilde{T}$ is a collection of closed subsets of $\tilde{X}$ and $\tilde{T}$ is contained in the usual subbase for $\tilde{X}$. Thus to show that $\tilde{T}$ generates the interval topology on $\tilde{X}$ we need only verify that all principal ideals of $\tilde{X}$ and principal dual ideals of $\tilde{X}$, which are not in $\tilde{T}$, are closed in the topology generated by $\tilde{T}$. This follows immediately from Proposition 5.5 parts (ii) and (iii).

To see that $\{d^{-1}[A] : A \in \tilde{T}_D\} = T_D$ and $\{d^{-1}[A] : A \in \tilde{T}_I\} = T_I$, note that $d^{-1}[\tilde{X}] = X$ and since d is a lattice inclusion, $d^{-1}[\tilde{X}_D(d(a))] = X_D(a)$ and $d^{-1}[\tilde{X}_I(d(a))] = X_I(a)$ for each $a \in X$.

PROPOSITION 5.8. The completion of X by cuts is a compactification of X; specifically, the mapping $d: X \to \tilde{X}$ is a topologically dense embedding.

PROOF. Since $\tilde{X}$ is complete, we know that it is compact (2.9). That d is continuous follows immediately from the facts that $\tilde{T}$ is a subbase for the closed subsets of $\tilde{X}$ and $\{d^{-1}[A]: A \in \tilde{T}\} = T$ (5.7) is a collection of closed subsets of X.

That d is injective follows from the fact that d is a lattice inclusion (2.2 (ii)).

To see that d is an embedding, we need only show that d is closed onto its image. Recall that d is a lattice inclusion and is (consequently) injective. Thus for each $a \in X$, $d[X_D(a)] = \tilde{X}_D(d(a)) \cap d[X]$ and $d[X_I(a)] = \tilde{X}_I(d(a)) \cap d[X]$; therefore, $d[X_D(a)]$ and $d[X_I(a)]$ are closed in d[X]. Hence d is an embedding since T is a subbase for the closed sets of X.

Finally, to see that d[X] is topologically dense in $\tilde{X}$, recall that we require that $X \neq \emptyset$. Let $A \in \tilde{X}$. If $\wedge \tilde{X} < A$, then $A \neq \emptyset$ and so there exists some $a \in A$. Since $d(a) \leq A$, it follows from Proposition 5.6 that $A \in \overline{d[X_I(a)]} \subseteq \overline{d[X]}$. If $\wedge \tilde{X} = A$, then let $a \in X$. Since $A = \wedge \tilde{X} \leq d(a)$, it follows again from Proposition 5.6 that $A \in \overline{d[X_D(a)]} \subseteq \overline{d[X]}$. Hence $\tilde{X} \subseteq d[X]$; i.e., d[X] is topologically dense in $\tilde{X}$.

We return now to the discussion of lattice superextensions.

PROPOSITION 5.9. $\tilde{T}$ is an admissible subbase for $\tilde{X}$.

PROOF. By Lemma 5.7, $\tilde{T}$ is a subbase for X. To see that $\tilde{T}$ is a T_1-subbase for $\tilde{X}$ let $A \in \tilde{X}$. By the definition of $\tilde{T}$, $\cap\{U \in \tilde{T}: A \in U\} = [\cap\{\tilde{X}_D(d(a)): A \leq d(a)\}] \cap [\cap\{\tilde{X}_I(d(a)): d(a) \leq A\}] \cap \tilde{X}$. Since $\cap\{\tilde{X}_D(d(a)): A \leq d(a)\} = \tilde{X}_D(A)$ (5.5 (ii)) and $\cap\{\tilde{X}_I(d(a)): d(a) \leq A\} = \cap\{\tilde{X}_I(d(a)): a \in A\} = \tilde{X}_I(A)$ (5.5 (iii)), it follows that

$$\cap\{U \in \tilde{T}: A \in U\} = \tilde{X}_D(A) \cap \tilde{X}_I(A) = \{A\}.$$

Finally, suppose that $A \in \tilde{X}$ and $U \in \tilde{T}$ such that $A \notin U$. We consider first the case that $U = \tilde{X}_D(d(a))$ for some $a \in X$. Since $A \notin U$, it follows that $A \not\leq d(a)$. Since $A = \vee\{d(x): x \in A\}$ (5.5 (i)) there is some $x \in A$ such that $d(x) \not\leq d(a)$. Thus $\tilde{X}_I(d(x)) \in \tilde{T}_I$ and it follows that $A \in \tilde{X}_I(d(a))$ and $\tilde{X}_I(d(a)) \cap \tilde{X}_D(d(a)) = \emptyset$. Since $x \in A$ if and only if $d(x) \leq A$, we may dualize the above argument in the case that $U \in \tilde{T}_I$.

Hence $\tilde{T}$ is a T_1-subbase for $\tilde{X}$ (2.11 (ii)). Since $\tilde{T}$ consists of ideals and dual ideals of X, it is an admissible subbase for X (4.2 (i)).

PROPOSITION 5.10. *If* $a, b \in X$, *then* $\tilde{X}_D(d(a)) \cap \tilde{X}_I(d(b)) = \emptyset$ *if and only if* $X_D(a) \cap X_I(b) = \emptyset$.

PROOF. Since $d[X_D(a)] \subseteq \tilde{X}_D(d(a))$ and $d[X_I(b)] \subseteq \tilde{X}_I(d(b))$, it is easily seen that $\tilde{X}_D(d(a)) \cap \tilde{X}_I(d(b)) = \emptyset$ implies that $X_D(a) \cap X_I(b) = \emptyset$. Conversely, if $X_D(a) \cap X_I(b) = \emptyset$, then since $b \not\leq a$ and d is a lattice inclusion, it follows that $d(b) \not\leq d(a)$. Hence $\tilde{X}_D(d(a)) \cap \tilde{X}_I(d(b)) = \emptyset$.

The following theorem will have several applications in this chapter and Chapters 6 and 8.

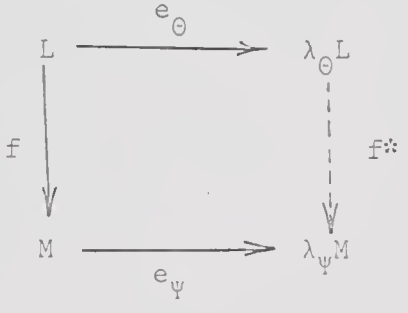
THEOREM 5.11. *Suppose that* L *and* M *are topological lattices which satisfy the following conditions*:

(i) $f: L \to M$ *is a continuous lattice inclusion*.

(ii) Θ *and* Ψ *are admissible subbases for* L *and* M, *respectively, such that* $\Theta_D = \{f^{-1}[V]: V \in \Psi_D\}$ *and* $\Theta_I = \{f^{-1}[V]: V \in \Psi_I\}$.

(iii) *If* $V_1, V_2 \in \Psi$ *such that* $V_1 \cap V_2 \neq \emptyset$, *then* $f^{-1}[V_1] \cap f^{-1}[V_2] \neq \emptyset$.

Then $\lambda_\Theta L$ *is iseomorphic with* $\lambda_\Psi M$ *via an iseomorphism* f^* *which is an extension of* f; i.e., *such that the following diagram commutes*.



PROOF. *Definition of* f^*: Suppose that $\mathcal{L} \in \lambda_\Theta L$. Clearly, $\mathcal{L}' = \{V \in \Psi: f^{-1}[V] \in \mathcal{L}\}$ is a linked system of Ψ. If $V \in \Psi$ and $\{V\} \cup \mathcal{L}'$ is linked, then by conditions (ii) and (iii), $\mathcal{L} \cup \{f^{-1}[V]\}$ is linked. Since $\mathcal{L}$ is a maximal linked system of Θ, $f^{-1}[V] \in \mathcal{L}$ and so $V \in \mathcal{L}'$. Thus $\{V \in \Psi: f^{-1}[V] \in \mathcal{L}\}$ is a maximal linked system of Ψ. We define $f^*: \lambda_\Theta L \to \lambda_\Psi M$ by $f^*(\mathcal{L}) = \{V \in \Psi: f^{-1}[V] \in \mathcal{L}\}$ for each $\mathcal{L} \in \lambda_\Theta L$.

f* is an extension of f; i.e., $e_\Psi \circ f = f^* \circ e_\Theta$: In view of condition (ii) we see that for each $x \in X$, $f^* \circ e_\Theta(x)$ $= f^*(\{W \in \Theta: x \in W\}) = f^*(\{f^{-1}[V]: x \in f^{-1}[V], V \in \Psi\})$ $= \{V \in \Psi: f(x) \in V\} = e_\Psi(f(x))$. Thus the diagram commutes.

f is surjective: For each $\mathcal{M} \in \lambda_\Psi M$ it follows from (ii) and (iii) that $\{f^{-1}[V]: V \in \mathcal{M}\}$ is a maximal linked system of Θ. By the definition of f*, $f^*(\{f^{-1}[V]: V \in \mathcal{M}\}) = \mathcal{M}$; therefore, f* is surjective.

f* is a lattice inclusion: To see that f* is injective suppose $\mathcal{J}, \mathcal{K} \in \lambda_\Theta L$ such that $\mathcal{J} \neq \mathcal{K}$. Thus it follows from (ii) that there are $V, V' \in \Psi$ with $f^{-1}[V] \in \mathcal{J}$, $f^{-1}[V'] \in \mathcal{K}$, and $f^{-1}[V] \cap f^{-1}[V'] = \emptyset$. By condition (iii), $V \cap V' = \emptyset$ and since, by the definition of f*, $V \in f^*(\mathcal{J})$ and $V' \in f^*(\mathcal{K})$, it follows that $f^*(\mathcal{J}) \neq f^*(\mathcal{K})$.

To see that f* is a lattice homomorphism, suppose that C is a non-void subset of $\lambda_\Theta L$, and let $\mathcal{L}_C = \vee C$. Let $V \in \Psi_D$. By the definition of f*, $V \in f^*(\mathcal{L}_C)$ if and only if $f^{-1}[V] \in \mathcal{L}_C$. Since $\mathcal{L}_C = \vee C$, $f^{-1}[V] \in \mathcal{L}_C$ if and only if $f^{-1}[V] \in \mathcal{L}$ for each $\mathcal{L} \in C$ (4.8). For each $\mathcal{L} \in C$, it follows from the definition of f* that $f^{-1}[V] \in \mathcal{L}$ if and only if $V \in f^*(\mathcal{L})$. Finally, by Proposition 4.8, $V \in f^*(\mathcal{L})$ for each $\mathcal{L} \in C$ if and only if $V \in \vee f^*[C]$. It follows from the above equivalences that $V \in f^*(\mathcal{L}_C)$ if and only if $V \in \vee f^*[C]$. Thus $f^*(\mathcal{L}_C) \cap \Psi_D = \vee f^*[C] \cap \Psi_D$ and consequently, $f^*(\mathcal{L}_C) = \vee f^*[C]$ (4.7).

A dual argument shows that f* preserves infima of $\lambda_\Theta L$. Hence f* is an injective lattice homomorphism; i.e., f* is a lattice inclusion (2.2 (ii)).

f* is continuous: It follows from the definition of f* that for each $f^{-1}[V] \in \Theta$ and each $\mathcal{L} \in \lambda_\Theta L$, $f^{-1}[V] \in \mathcal{L}$ if and only if $V \in f^*(\mathcal{L})$. Thus for each $V \in \Psi$,

$$f^{*-1}[V^+] = \{\mathcal{L} \in \lambda_\Theta L: f^*(\mathcal{L}) \in V^+\}$$
$$= \{\mathcal{L} \in \lambda_\Theta L: V \in f^*(\mathcal{L})\} \quad (2.14)$$
$$= \{\mathcal{L} \in \lambda_\Theta L: f^{-1}[V] \in \mathcal{L}\} = (f^{-1}[V])^\tau$$

which is a member of the subbase for the closed sets of $\lambda_\Theta L$. Since $\{V^+: V \in \Psi\}$ is a subbase for the closed subsets of $\lambda_\Psi M$, it follows that f* is continuous.

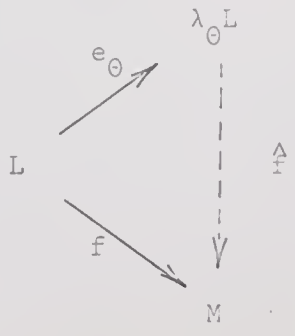
f* is a closed map: For each $V \in \Psi$, it follows from the above equality and the injectivity of f* that

$$f^*[(f^{-1}[V])^+] = f^*[f^{*-1}[V^+]] = V^+.$$

By condition (ii), $\{f^{-1}[V]: V \in \Psi\} = \Theta$ and so $\{(f^{-1}[V])^+: V \in \Psi\} = \Theta^+$ is a subbase for the closed subsets of $\lambda_\Theta L$. It now follows that the image under f* of each subbasic closed set in $\lambda_\Theta L$ is a subbasic closed set in $\lambda_\Psi M$ and since f* is injective, f* is a closed map.

Thus f* satisfies all of the conditions of Definition 2.10, and so f* is an iseomorphism of $\lambda_\Theta L$ onto $\lambda_\Psi M$.

PROPOSITION 5.12. If L, M, f, Θ, and Ψ satisfy the hypotheses of Theorem 5.11 and M is complete, then M is iseomorphic with $\lambda_\Theta L$ via an iseomorphism $\hat{f}$ which makes the following diagram commutative:



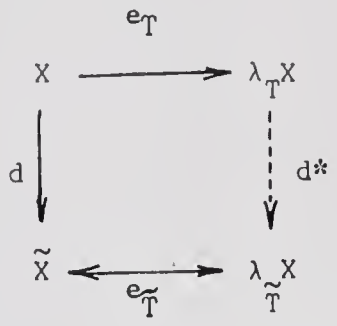
PROOF. In view of Theorem 5.11, we need only show that $e_\Psi: M \to \lambda_\Psi M$ is an iseomorphism; i.e., that e_Ψ is surjective. If $\mathcal{L} \in \lambda_\Psi M$, then by Proposition 4.5 (ii), $\mathcal{L}$ is centered. Since M is compact and $\mathcal{L}$ consists of closed subsets of M, it follows that $\cap\mathcal{L} \neq \emptyset$. By Proposition 2.16 (iv), $\cap\mathcal{L}$ is a singleton; i.e., $\cap\mathcal{L} = \{a\}$ for some $a \in M$. It follows that $\mathcal{L} = e_\Psi(a)$.

Thus if f^* is the iseomorphism defined in Theorem 5.11, then $\hat{f} = e_\Psi^{-1} \circ f^*: \lambda_\Theta L \to M$ is an iseomorphism with the property that $\hat{f} \circ e_\Theta = e_\Psi^{-1} \circ f^* \circ e_\Theta = e_\Psi^{-1} \circ e_\Psi \circ f = f$.

THEOREM 5.13. If X is a topological lattice, then the completion of X by cuts is iseomorphic with the superextension of X with respect to the usual subbase T; i.e., $\tilde{X}$ is iseomorphic with $\lambda_T X$.

PROOF. Clearly X and $\tilde{X}$ are topological lattices and d is a continuous lattice inclusion (5.4 and 5.8). T and $\tilde{T}$ are admissible subbases for X and $\tilde{X}$, respectively (4.4 and 5.9), and $\{d^{-1}[A]: A \in T_D\} = T_D$ and $\{d^{-1}[A]: A \in T_I\} = T_I$ (5.7). Suppose that V_1, $V_2 \in T$ such that $V_1 \cap V_2 \neq \emptyset$. If V_1 and V_2 are both in $\tilde{T}_D$ or both in $\tilde{T}_I$, then it follows from the definition of $\tilde{T}$ (5.7) that $d^{-1}[V_1] \cap d^{-1}[V_2] \neq \emptyset$. In the remaining case we may assume that $V_1 \in \tilde{T}_D$ and $V_2 \in \tilde{T}_I$. It now follows from the definition of $\tilde{T}$ (5.7) and Proposition 5.10 that $d^{-1}[V_1] \cap d^{-1}[V_2] \neq \emptyset$.

Thus X, $\tilde{X}$, d, T, and $\tilde{T}$ satisfy the hypotheses of Theorem 5.11, and since $\tilde{X}$ is complete, it follows from Proposition 5.12 that $\tilde{X}$ is iseomorphic with $\lambda_T X$ and the following diagram commutes.



If we restrict our attention to the case where X is totally ordered, then T becomes a weakly-normal subbase (4.12), and the completion by cuts ($\lambda_T X$) is Hausdorff (2.18 (ii)). The question arises as to how we can characterize the locally compact, totally ordered spaces for which the Alexandroff one-point compactification (ωX) is the same as the completion of X by cuts. Before answering this question we need a few preliminary results.

DEFINITION 5.14. For each $A \subseteq X$, $D(A)^c = \{x \in X: D(A) \subseteq X_D(x)\}$.

PROPOSITION 5.15. For any subset $A \subseteq X$, D(A) and $D(A)^c$ are closed subsets of X. Furthermore, D(A) (dually, $D(A)^c$) is either void or decreasing (dually, increasing).

PROOF. Let $A \subseteq X$. That $D(A)$ is closed follows from the fact that $D(A) = \bigcap\{X_D(a): A \subseteq X_D(a)\}$ (5.3). By Proposition 2.4 (i), $D(A)$ is either void or an ideal of X; i.e., $D(A)$ is either void or decreasing. To see that $D(A)^c$ is a closed subset of X, we will show that $D(A)^c = \bigcap\{X_I(a): a \in A\}$. Since $A \subseteq D(A)$, it follows from the definition of $D(A)^c$ that for each $a \in A$ and each $x \in D(A)^c$, $a \leq x$ so that $x \in X_I(a)$. Thus $D(A)^c \subseteq \bigcap\{X_I(a): a \in A\}$. To see the other inclusion, suppose that $x \in X$ such that $x \notin D(A)^c$. Thus $A \not\subseteq X_D(x)$ so there is some $a_0 \in A$ with $a_0 \not\leq x$. Hence $x \notin X_I(a_0)$ and so $x \notin \bigcap\{X_I(a): a \in A\}$; therefore, $D(A)^c = \bigcap\{X_I(a): a \in A\}$ is a closed subset of X. It now follows from Proposition 2.4 (i) that $D(A)^c$ is either void or increasing.

LEMMA 5.16. Let X be a totally ordered topological lattice.

(i) For each $A \subseteq X$, $D(A) \cap D(A)^c \neq \emptyset$ if and only if $\vee A$ exists in X; moreover, if $\vee A$ exists in X, then $D(A) \cap D(A)^c = \{\vee A\}$.

(ii) For each $x \in X$, $\vee D(\{x\}) = x$.

(iii) For each $A \subseteq X$, if $\vee_X D(A)$ exists, then $\wedge_X D(A)^c$ exists and $\vee_X D(A) = \wedge_X D(A)^c$.

(iv) For each $A \subseteq X$, if any of $\vee A$, $\vee D(A)$, or $\wedge D(A)^c$ exists in X, then they all exist and are equal.

(v) For each $A \subseteq X$, if $\wedge d[A] = d(x)$ for some $x \in X$, then $\wedge A = x$.

PROOF. (i) Let $A \subseteq X$. Suppose that $b = \vee A$ exists in X. Thus $A \subseteq X_D(b)$ so that $D(A) \subseteq X_D(b)$ and hence $b \in D(A)^c$. If $a \in X$ such that $A \subseteq X_D(a)$, then $b = \vee A \in X_D(a)$. Thus $b \in \bigcap\{X_D(a): A \subseteq X_D(a)\} = D(A)$.

Hence $b \in D(A) \cap D(A)^c$. If $x \in X$ such that $x \in D(A) \cap D(A)^c$, then since $x \in D(A)^c$ and $b \in D(A)$, it follows from the definition of $D(A)^c$ that $b \in D(A) \subseteq X_D(x)$ and so $b \leq x$. In the same way since $b \in D(A)^c$ and $x \in D(A)$, it follows that $x \in X_D(b)$; therefore, $x \leq b$. Thus $x = b$ and $D(A) \cap D(A)^c = \{\vee A\}$.

Conversely, suppose now that $D(A) \cap D(A)^c \neq \emptyset$ and let $b \in D(A) \cap D(A)^c$. Thus $A \subseteq D(A) \subseteq X_D(b)$ so that $a \leq b$ for each $a \in A$. If $x \in X$ such that $x < b$, then since $b \notin X_D(x)$ and $b \in D(A)$, it follows from the definition of $D(A)$ (5.3) that $A \not\subseteq X_D(x)$. Thus there is some $a \in A$ such that $a \notin X_D(x)$; i.e., $a \not\leq x$. It now follows that $b = \vee A$. Thus $D(A) \cap D(A)^c = \{\vee A\}$.

(ii) Since for each $x \in X$, $D(\{x\}) = X_D(x)$, it is clear that $x = \vee D(\{x\})$.

(iii) Let $A \subseteq X$ such that $\vee D(A)$ exists in X. Let $b = \vee D(A)$. Thus $D(A) \subseteq X_D(b)$ and by the definition of $D(A)^c$, $b \in D(A)^c$. Since $D(A)^c$ is increasing (5.15), it follows that $X_I(b) \subseteq D(A)^c$. Recall that $D(A)$ is closed in the interval topology (5.15) and hence in the order topology [9 ; Theorem 12]; hence, $b = \vee D(A) \in D(A)$. It follows that for each $x \in D(A)^c$, $b \leq x$ and so $x \in X_I(b)$; therefore, $D(A)^c \subseteq X_I(b)$. Thus $D(A)^c = X_I(b)$ and so $\wedge D(A)^c = b = \vee D(A)$.

(iv) Let $A \subseteq X$ and suppose that $\vee A$ exists in X. Let $a = \vee A$. By (i) $a \in D(A)^c$ so that $D(A) \subseteq X_D(a)$. Also by (i), $a \in D(A)$ and since $D(A)$ is decreasing (5.15), it follows that $X_D(a) \subseteq D(A)$. Thus $D(A) = X_D(a) = D(\{a\})$. By (ii), $\vee D(A)$ exists in X and $\vee D(A) = a$. Now using (iii) we see that $\wedge D(A)^c$ exists in X and $\wedge D(A)^c = a$.

Suppose now that $\vee D(A)$ exists in X. By (iii), $\wedge D(A)^c$ exists and $\wedge D(A)^c = \vee D(A)$. Thus we need only show that $\vee A$ exists and $\vee A = \vee D(A)$. Let $a = \vee D(A)$. Since $D(A)$ is closed in the interval topology (5.15), it is also closed in the order topology [9 ; Theorem 12] and so $a = \vee D(A) \in D(A)$. Since $D(A)^c$ is also closed in the interval topology (5.15) and hence in the order topology, it follows that $a = \wedge D(A)^c \in D(A)^c$. Thus $a \in D(A) \cap D(A)^c$ and by (i) $\vee A$ exists in X and $a = \vee A$.

Finally suppose that $\wedge D(A)^c$ exists in X and let $a = \wedge D(A)^c$. Since $D(A)^c$ is closed, it follows as before that $a = \wedge D(A)^c \in D(A)^c$. Thus $D(A) \subseteq X_D(a)$. Now if $x \in X$ such that $A \subseteq X_D(x)$, then by the definition of $D(A)$, $D(A) \subseteq X_D(x)$ so that $x \in D(A)^c$. Thus $a \leq x$ and so $a \in X_D(x)$. Hence $a \in \cap\{X_D(x): A \subseteq X_D(x)\} = D(A)$. Thus $D(A) \cap D(A)^c \neq \emptyset$. By (i) $\vee A$ exists in X and $a = \vee A$. By a previous case $\vee D(A)$ exists and $\vee D(A) = a = \vee A = \wedge D(A)^c$.

(v) Suppose $A \subseteq X$ such that $\wedge d[A] = d(x)$ for some $x \in X$. For each $a \in A$, $d(x) \leq d(a)$ and so $x \leq a$ since d is a lattice inclusion. If $c \in X$ such that $x < c$, then since d is injective, $d(x) < d(c)$ and there is some $d(a) \in d[A]$ such that $d(x) \leq d(a)$ and $d(c) \not\leq d(a)$. Since $\tilde{X}$ is totally ordered (4.12 and 5.13) it must be the case that $d(a) < d(c)$. Thus since d is injective, $x \leq a < c$. It follows that $x = \wedge A$.

The following lemma is well known and so its proof will be omitted.

LEMMA 5.17. If H is any Hausdorff locally compact space, and if Y and Z are two Hausdorff compactifications of H with $|Y - H| = |Z - H| = 1$, then Z and Y are homeomorphic.

The following theorem characterizes those situations in which the completion by cuts is the Alexandroff one-point compactification.

THEOREM 5.18. Suppose that X is a locally compact, totally ordered topological lattice. Then in order for the completion by cuts of X ($\tilde{X}$) to be homeomorphic with the Alexandroff one-point compactification of X (ωX) it is necessary and sufficient that there exists exactly one subset $A \subset X$ such that $\vee A$ and $\wedge (X - A)$ do not exist in X.

PROOF. Necessity: If there does not exist $B \subset X$ such that $\vee B$ does not exist in X, then for each $B \subset X$, $\vee B = \vee D(B) = b$ for some $b \in X$ (5.16 (iv)). Thus $D(B) \neq \emptyset$, and since D(B) is a closed ideal of X (2.4 (i)), it follows that $D(B) = d(b)$. Therefore $X = \tilde{X}$ is compact, and since $X \neq \omega X$, $\tilde{X} \neq \omega X$ -- a contradiction. Thus there is some $B \subset X$ such that $\vee B$ does not exist in X. Let $A = D(B)$. Since A is decreasing (5.15) and X is totally ordered, it follows that $X - A = D(A)^c$. Since $D(A) = D(D(B)) = D(B)$, it follows that $D(A)^c = D(B)^c$. Thus by Lemma 5.16 (iv), $\vee A$ and $\wedge (X - A)$ do not exist in X and so at least one such subset of X exists. Suppose now that there is another set $C \subset X$ such that $\vee C$ and $\wedge (X - C)$ do not exist in X. Recall that D(C) and A are members of $\tilde{X}$ and that $\tilde{X}$ is homeomorphic with ωX. If $D(C) \in d[X]$, then there is

some $x \in X$ such that $D(C) = d(x)$. Since $d(x) = D(\{x\})$, it follows that $\vee D(C) = x$ (5.16 (ii)). Hence $\vee C = x$ (5.16 (iv)), but this contradicts the fact that $\vee C$ does not exist in X. Thus $D(C) \notin d[X]$. By a similar argument, using the fact that $\vee A$ does not exist in X, we see that $A \notin d[X]$. Now since $|\tilde{X} - d[X]| = |\omega X - X| = 1$ and $D(C), A \in \tilde{X} - d[X]$, it follows that $D(C) = A$. If $C \neq D(C)$, then since $C \subset D(C)$, there is some $x \in D(C) - C$, and since $\vee D(C)$ does not exist in X, $d(x) < D(C)$ in $\tilde{X}$. Let $G = d[X - C]$. Then $d(x) \in G$ so that G is a non-void subset of X. Since $\tilde{X}$ is complete, $\wedge G$ exists in $\tilde{X}$ and $\wedge G \leq d(x)$. Thus $\wedge G \neq D(C)$, so it must be the case that $\wedge G \in d[X]$ and there is some $z \in X$ such that $\wedge G = d(z)$. Hence $z = \wedge (X - C)$ (5.16 (v)); i.e., $\wedge (X - C)$ exists in X--a contradiction. Therefore, $C = D(C) = A$, and A is the unique subset of X for which $\vee A$ and $\wedge (X - A)$ do not exist in X.

Sufficiency: Let $A \subseteq X$ be the unique set such that $\vee A$ and $\wedge (X - A)$ do not exist in X. Thus $\vee D(A)$ does not exist in X (5.16 (iv)), and since $X - D(A) = D(A)^c$, $\wedge (X - D(A))$ does not exist in X (5.16 (iv)). By the uniqueness of A, $A = D(A)$ and so $D(A) \neq D(\{x\})$ for all $x \in X$ (5.16 (ii)); i.e., $D(A) = A \in \tilde{X} - d[X]$. If $B \in \tilde{X}$ such that $B \notin d[X]$, then $\vee B$ does not exist in X, for suppose that there is some $b \in X$ such that $\vee B = b$. Then since B is a closed decreasing subset of X (5.15), it follows that $b \in B$ and $X_D(b) = B$; therefore, $B = D(\{b\}) = d(b) \in d[X]$ --a contradiction. Finally, since $\vee B$ does not exist in X and $B = D(B)$, it follows that $\wedge (X - B) = \wedge D(B)^c$ does not exist in X (5.16 (iv)). Hence by the uniqueness of A, $A = B$.

Thus $\tilde{X} - d[X] = \{A\}$ and since $\tilde{X}$ is a Hausdorff compactification of the locally compact Hausdorff space X formed by adjoining one point to X, $\tilde{X}$ is homeomorphic with ωX (5.17).

In view of the fact that the open n-cell in Euclidean n-space can be characterized as the product of n-copies of the open interval (0, 1), it follows from Theorem 5.13 and the Product Theorem (3.4), that the closed n-cell (i.e., the n-cell with boundary attached) is a de Groot compactification of the open disc. Likewise, since Hilbert space can be characterized as the countably infinite product of the open interval (0, 1) [3], it follows that the Hilbert cube is a de Groot compactification of Hilbert space. (See Example 8.3 for details.)

6. TOPOLOGICAL LATTICE COMPLETIONS

Recall that Theorem 5.13 provided a superextension characterization of the completion by cuts. The purpose of this chapter is to obtain algebraic characterizations of other lattice superextensions. This is done via the notion of topological lattice completions.

Throughout this chapter X will denote a topological lattice. The topological lattice completions of X are complete topological lattices having X as a sublattice and a subspace. We show that all lattice superextensions of X are actually topological lattice completions of X (Theorem 6.22). Conversely, all topological lattice completions of X which satisfy a certain density condition are shown to be lattice superextensions of X (Theorem 6.20).

Throughout this chapter Y will denote a complete topological lattice (i.e., a complete lattice equipped with the interval topology) and all subbases considered are assumed to be subbases for closed sets.

NOTATION 6.1. Let Z be a non-void subset of Y.

(i) $Z^0 = Z$.

(ii)

$$Z^{1a} = \{y \in Y: y = \begin{cases} \vee_Y\{z \in Z: z \leq y\}, \text{ if } \{z \in Z: z \leq y\} \neq \emptyset, \text{ or} \\ \wedge_Y Z, \text{ if } \{z \in Z: z \leq y\} = \emptyset \end{cases}\}$$

$$Z^{1b} = \{y \in Y: y = \begin{cases} \wedge_Y\{z \in Z: y \leq z\}, \text{ if } \{z \in Z: y \leq z\} \neq \emptyset, \text{ or} \\ \vee_Y Z, \text{ if } \{z \in Z: y \leq z\} = \emptyset \end{cases}\}$$

$$Z^1 = Z^{1a} \cap Z^{1b}.$$

(iii) $Z^{2a} = (Z^{1a})^{1b}$

$Z^{2b} = (Z^{1b})^{1a}$

$Z^{2} = Z^{2a} \cap Z^{2b}$.

(iv) $\emptyset^{0} = \emptyset^{1a} = \emptyset^{1b} = \emptyset^{2a} = \emptyset^{2b} = \emptyset$.

It should be noted that Definition 6.1 can be easily extended to obtain Z^{α} for each ordinal α, and if we define the "lattice closure of Z in Y" to be Z^{α} where α is the least ordinal such that $Z^{\alpha} = Z^{\alpha+1}$, then the map which takes Z onto its lattice closure in Y can be shown to be a Moore closure operator, or completion-operator on Y as in Definition 5.1. See Appendix II for the details.

PROPOSITION 6.2. Suppose that $A \subseteq B \subseteq Y$. Then

(i) $A^{1a} \subseteq B^{1a}$

(ii) $A^{1b} \subseteq B^{1b}$

(iii) $A^{1} \subseteq B^{1}$

(iv) $A \subseteq A^{1} \subseteq A^{1a} \cup A^{1b} \subseteq A^{2}$.

PROOF. (i) If $y \in A^{1a}$, then $\{a \in A: a \leq y\} \subseteq \{b \in B: b \leq y\}$ and $y = \vee \{a \in A: a \leq y\} \leq \vee \{b \in B: b \leq y\} \leq y$; i.e., $y \in B^{1a}$.

(ii) Dualize the argument for (i).

(iii) This is immediate from (i) and (ii) above and the definitions of A^{1} and B^{1} (6.1 (ii)).

(iv) The proof is immediate if $A = \emptyset$ (6.1 (iv)); so assume that $A \neq \emptyset$.

That $A \subseteq A^1$ follows from the facts that for each $w \in A$, $w = \vee \{a \in A: a \leq w\} \in A^{1a}$ and $w = \wedge \{a \in A: w \leq a\} \in A^{1b}$. Clearly, $A^1 \subseteq A^{1a} \cup A^{1b}$ (6.1 (ii)).

To see that $A^{1a} \subseteq A^2$, let $w \in A^{1a}$. Thus $w = \wedge \{z \in A^{1a}: w \leq z\} \in (A^{1a})^{1b} = A^{2a}$. Finally, since $A \subseteq A^{1b}$, it follows from (ii) above that $w \in A^{1a} \subseteq (A^{1b})^{1a} = A^{2b}$; therefore, $w \in A^{2a} \cap A^{2b} = A^2$. That $A^{1b} \subseteq A^2$ follows dually. Thus $A \subseteq A^1 \subseteq A^{1a} \cup A^{1b} \subseteq A^2$.

PROPOSITION 6.3. *Suppose that* $A \subseteq Y$.

(i) *If* $y \in A^{1a}$ *and* $A \cap Y_D(y) \neq \emptyset$, *then* $Y_I(y) = \cap\{Y_I(a): a \in A$ *and* $a \leq y\}$.

(ii) *If* $y \in A^{1b}$ *and* $A \cap Y_I(y) \neq \emptyset$, *then* $Y_D(y) = \cap\{Y_D(a): a \in A$ *and* $y \leq a\}$.

PROOF. Both parts (i) and (ii) follow immediately from the definitions.

PROPOSITION 6.4. *Suppose that* $A \subseteq Y$.

(i) *If* A *contains the suprema of all non-void finite subsets of* A *and* $y \in A^{1a}$ *such that* $A \cap Y_D(y) \neq \emptyset$, *then* $y \in \overline{(A \cap Y_D(y))}^Y$.

(ii) *If* A *contains the infima of all non-void finite subsets of* A *and* $y \in A^{1b}$ *such that* $A \cap Y_I(y) \neq \emptyset$, *then* $y \in \overline{(A \cap Y_I(y))}^Y$.

PROOF. (i) Let A satisfy the hypotheses of (i) and suppose that $y \in A^{1a}$ such that $A \cap Y_D(y) \neq \emptyset$. To see that $y \in \overline{(A \cap Y_D(y))}^Y$, let W be any basic open set of Y containing y.

If $W = Y - \cup\{Y_I(z_i): i = 1, \cdots, n\}$ for some finite set $\{z_1, \cdots, z_n\} \subseteq Y$, then since for each i, $z_i \not\leq y$, it follows that $Y_D(y) \cap (\cup\{Y_I(z_i): i = 1, \cdots, n\}) = \emptyset$ and so $(A \cap Y_D(y)) \cap (\cup\{Y_I(z_i): i = 1, \cdots, n\}) = \emptyset$. Hence $A \cap Y_D(y) \subseteq W$ and $W \cap (A \cap Y_D(y)) \neq \emptyset$.

If $W = Y - \cup\{Y_D(w_j): j = 1, \cdots, m\}$ for some finite set $\{w_1, \cdots, w_m\} \subseteq Y$, then for each $j = 1, \cdots, m$, $y \not\leq w_j$. Since $y = \vee\{a \in A: a \leq y\}$, it is possible to pick, for each $j \in \{1, \cdots, m\}$, some $x_j \in A$ with the property that $x_j \leq y$ and $x_j \not\leq w_j$. Let $a = \vee\{x_j: j = 1, \cdots, m\}$. By the hypotheses on A and since each $x_j \in A$, it follows that $a \in A$ and $a \leq y$; i.e., $a \in A \cap Y_D(y)$. If $a \notin W$, then there is some $j \in \{1, \cdots, m\}$ with $a \in Y_D(w_j)$; therefore, $x_j \leq a \leq w_j$ and $x_j \leq w_j$--a contradiction. Thus $a \in W$ and so $W \cap (A \cap Y_D(y)) \neq \emptyset$.

In the remaining case $W = U \cap V$ where $U = Y - \cup\{Y_D(w_j): j = 1, \cdots, m\}$ and $V = Y - \cup\{Y_I(z_i): i = 1, \cdots, n\}$ for some finite set $\{w_1, \cdots, w_m, z_1, \cdots, z_n\} \subseteq Y$. By an argument analogous to the one above, there is some $a \in U \cap (A \cap Y_D(y))$ and since $A \cap Y_D(y) \subseteq V$, it follows that $a \in W \cap (A \cap Y_D(y))$.

Since W was arbitrarily chosen, it follows that $y \in \overline{(A \cap Y_D(y))}^Y$.

(ii) Dualize the argument for (i).

PROPOSITION 6.5. *If* $i: X \to Y$ *is a lattice inclusion and if* $i[X]^l = Y$, *then* $\vee i[X] = \vee Y$ *and* $\wedge i[X] = \wedge Y$.

PROOF. Clearly $\vee Y \in Y = i[X]^1 \subseteq i[X]^{1a}$ (6.2 (iv)). Thus $\vee Y = \vee \{i(x) \in i[X]: i(x) \leq \vee Y\} = \vee \{i(x) \in i[X]\} = \vee i[X]$. The other equality is proved dually.

NOTATION 6.6. If $i: X \to Y$ is a lattice inclusion such that $Y = i[X]^1$, then we let

$$\hat{T} = \{Y\} \cup \{Y_D(i(x)): x \in X\} \cup \{Y_I(i(x)): x \in X\}.$$

PROPOSITION 6.7. If $i: X \to Y$ is a lattice inclusion such that $i[X]^1 = Y$, then $\hat{T}$ is an admissible subbase for Y.

PROOF. Clearly $\hat{T}$ consists of ideals and dual ideals of Y. Since $Y = i[X]^1$, it follows from Proposition 6.3 that each member of the usual subbase for Y is an intersection of members of $\hat{T}$; therefore $\hat{T}$ is a subbase for the closed subsets of Y.

To see that $\hat{T}$ is a T_1-subbase for Y, suppose that $z \in Y$. Then since $z \in i[X]^1 = i[X]^{1a} \cap i[X]^{1b}$, it follows that

$$\cap\{T \in \hat{T}: z \in T\} = (\cap\{Y_D(i(x)): x \in X \text{ and } z \leq i(x)\}) \cap (\cap\{Y_I(i(x)): x \in X \text{ and } i(x) \leq z\}) \cap Y$$
$$= Y_D(z) \cap Y_I(z) \text{ (6.3)} = \{z\}.$$

Finally suppose that $z \in Y$ and $T \in \hat{T}$ such that $z \notin T$. Clearly $T \neq Y$. If $T \in \hat{T}_D$, then there is some $t \in X$ such that $T = Y_D(i(t))$. Since $z \not\leq i(t)$ and $z \in i[X]^1 \subseteq i[X]^{1a}$ (6.1 (ii)), there is some $x \in X$ such that $i(x) \leq z$ and $i(x) \not\leq i(t)$. Thus $z \in Y_I(i(x))$, $Y_I(i(x)) \in \hat{T}_I$, and $Y_D(i(t)) \cap Y_I(i(x)) = \emptyset$. If $T \in \hat{T}_I$, then a dual argument shows that

there is some $Y_D(i(x)) \in \hat{T}_D$ with $z \in Y_D(i(x))$ and $T \cap Y_D(i(x)) = \emptyset$. By Definition 2.11 (ii), $\hat{T}$ is a T_1-subbase for Y. Hence $\hat{T}$ is an admissible subbase for Y (4.2 (i)).

PROPOSITION 6.8. *If* i: X → Y *is a lattice inclusion such that* $Y = i[X]^1$, *then* $\{i^{-1}[T]: T \in \hat{T}_D\} = T_D$ *and* $\{i^{-1}[T]: T \in \hat{T}_I\} = T_I$ *where* T *is the usual subbase for* X.

PROOF. Clearly $T_D \subseteq \{i^{-1}[T]: T \in \hat{T}_D\}$ and $T_I \subseteq \{i^{-1}[T]: T \in \hat{T}_I\}$. To see the other containments, recall that i is a lattice inclusion; therefore, for each $a \in X$, $i^{-1}[Y_D(i(a))] = \{x \in X: i(x) \leq i(a)\}$ $= \{x \in X: x \leq a\} = X_D(a) \in T_D$ and dually, $i^{-1}[Y_I(i(a))] = X_I(a) \in T_I$.

PROPOSITION 6.9. *Suppose* i: X → Y *is a lattice inclusion such that* $i[X]^1 = Y$. *If* $S, T \in \hat{T}$, *then* $i^{-1}[S] \cap i^{-1}[T] \neq \emptyset$ *whenever* $S \cap T \neq \emptyset$.

PROOF. Suppose $S, T \in \hat{T}$ such that $S \cap T \neq \emptyset$. If S and T are both in $\hat{T}_D$ or both in $\hat{T}_I$, then the conclusion is immediate from the definition of $\hat{T}$ (6.6) and the fact that i is a lattice inclusion. For the remaining case we may assume that there exist t and s in X such that $T = Y_I(i(t))$ and $S = Y_D(i(s))$. Since $S \cap T \neq \emptyset$, we see that $i(t) \leq i(s)$ and since i is a lattice inclusion, $t \leq s$. Thus $i^{-1}[S] \cap i^{-1}[T] = X_D(s) \cap X_I(t) \neq \emptyset$.

DEFINITION 6.10. Suppose that X and Y are topological lattices, Y is complete, and i: X → Y is a function. Y will be called a *topological*

lattice completion of X with respect to i whenever the following two conditions are satisfied:

(i) $Y = i[X]^1$ or $Y = i[X]^2$.

(ii) i is a continuous lattice inclusion.

PROPOSITION 6.11. If $i: X \to Y$ is a lattice inclusion and $i[X]^1 = Y$, then Y is a topological lattice completion of X with respect to i; i.e., the map i is continuous.

PROOF. Since $\hat{T}$ is a subbase for Y (6.7), and $\{i^{-1}[T]: T \in \hat{T}\}$ is the usual subbase for X (6.8), it follows immediately that i is continuous.

PROPOSITION 6.12. If Y is a topological lattice completion of X with respect to $i: X \to Y$, then i is an iseomorphism onto its image; i.e., i is an embedding.

PROOF. In view of the definitions of topological lattice completions (6.10) and lattice inclusions (2.2 (ii)), we need only show that the restriction $i: X \to i[X]$ is a closed map. For each $a \in X$,

$i[X_D(a)] = \{i(x): x \in X \text{ and } x \leq a\}$.

$= \{i(x): x \in X \text{ and } i(x) \leq i(a)\}$ (since i is a lattice inclusion).

$= i[X] \cap Y_D(i(a))$.

Dually, for each $a \in X$, $i[X_I(a)] = i[X] \cap Y_I(i(a))$. Thus the image of every member of T is a closed set in i[X], and since i is injective, it follows that the image of each closed set in X is closed in i[X].

THEOREM 6.13. If $i: X \to Y$ is a lattice inclusion and $i[X]^1 = Y$, then Y is iseomorphic with the completion of X by cuts.

PROOF. By Proposition 6.11, the function $i: X \to Y$ is a continuous lattice inclusion. $\mathcal{T}$ and $\hat{\mathcal{T}}$ are admissible subbases for X and Y, respectively, such that $\{i^{-1}[T]: T \in \hat{\mathcal{T}}_D\} = \mathcal{T}_D$ and $\{i^{-1}[T]: T \in \hat{\mathcal{T}}_I\} = \mathcal{T}_I$ (6.7, 6.8, and 4.4). If $T, S \in \hat{\mathcal{T}}$ such that $T \cap S \neq \emptyset$, then $i^{-1}[T] \cap i^{-1}[S] \neq \emptyset$ (6.9). Thus X, Y, i, $\mathcal{T}$, and $\hat{\mathcal{T}}$ satisfy the hypotheses of Theorem 5.11, and since Y is compact it follows from Proposition 5.12 that Y is iseomorphic with $\lambda_{\mathcal{T}}X$, the completion of X by cuts (5.13).

REMARK 6.14. It follows from Theorem 6.13 that for any topological lattice X, there is only one completion Y of X for which $Y = X^1$. This occurs just when X^1 is a complete lattice. In the case where $X \subseteq Y$ and X^2, but not X^1, is a complete sublattice of Y, the possibilities are not so limited. In fact, in this case a lattice inclusion need not be continuous (see Example 8.6, Part I); however, for the remainder of this chapter we will consider only topological lattice completions of X.

In order to use Theorem 5.11 (as was done in the proof of Theorem 6.13) to characterize one of these "larger" topological lattice completions as a lattice superextension, it is necessary to obtain a subbase which satisfies certain conditions. Namely, if Y is a

topological lattice completion of X with respect to i, then we must choose some subbase Σ for Y such that

(i) Σ is an admissible subbase for Y,

(ii) $\{i^{-1}[S]: S \in \Sigma\}$ is an admissible subbase for X, and

(iii) if $S, T \in \Sigma$ with $S \cap T \neq \emptyset$, then $i^{-1}[S] \cap i^{-1}[T] \neq \emptyset$.

Keeping in mind conditions (i) and (ii) we make the following definition.

NOTATION 6.15. If Y is a topological lattice completion of X with respect to i, then

$$\Sigma^{(i)} = \{Y\} \cup \{Y_D(y): y \in (i[X]^{1a} - i[X]^1) \cup i[X]\}$$
$$\cup \{Y_I(y): y \in (i[X]^{1b} - i[X]^1) \cup i[X]\}.$$

PROPOSITION 6.16. *If Y is a topological lattice completion of X with respect to i and $Y = i[X]^1$, then $\Sigma^{(i)} = \hat{T}$.*

PROOF. This follows immediately from the fact that when $Y = i[X]^1$, then $i[X]^1 = i[X]^{1a} = i[X]^{1b}$ and so $i[X]^{1a} - i[X]^1 = i[X]^{1b} - i[X]^1 = \emptyset$.

PROPOSITION 6.17. *If Y is a topological lattice completion of X with respect to i, then*

(i) *$\Sigma^{(i)}$ is an admissible subbase for Y, and*

(ii) *$\{i^{-1}[S]: S \in \Sigma^{(i)}\}$ is an admissible subbase for X.*

PROOF. (i) Clearly $\Sigma^{(i)}$ consists of principal ideals and principal dual ideals of Y. In order to see that $\Sigma^{(i)}$ is a subbase for Y, we need only show that all principal ideals of Y and principal dual ideals of Y which are not in $\Sigma^{(i)}$ are closed in the topology on Y which has $\Sigma^{(i)}$ as a subbase. If $Y_D(y)$ is a principal ideal of Y and $Y_D(y) \notin \Sigma^{(i)}$, then it follows from the definition of $\Sigma^{(i)}$ that either

(a) $y \in i[X]^1$, or

(b) $y \in i[X]^2 - (i[X]^{1a} \cup i[X]^{1b})$.

Dually, if $z \in Y$ such that $Y_I(z) \notin \Sigma^{(i)}$, then either

(c) $z \in i[X]^1$, or

(d) $z \in i[X]^2 - (i[X]^{1a} \cup i[X]^{1b})$.

(a) If $y \in i[X]^1$, then $y \in i[X]^{1b}$, and so by Proposition 6.3 (ii),

$$Y_D(y) = \bigcap\{Y_D(i(x)): x \in X \text{ and } y \leq i(x)\}.$$

(c) If $z \in i[X]^1$, then $z \in i[X]^{1a}$, and by Proposition 6.3 (i),

$$Y_I(z) = \bigcap\{Y_I(i(x)): x \in X \text{ and } i(x) \leq z\}.$$

Thus for each $w \in i[X]^1$, $Y_I(w)$ and $Y_D(w)$ are closed in the topology generated by $\Sigma^{(i)}$. Hence in view of the definition of $\Sigma^{(i)}$ (6.15), for each $w \in i[X]^{1a} \cup i[X]^{1b}$, $Y_D(w)$ and $Y_I(w)$ are closed in the topology generated by $\Sigma^{(i)}$.

(b) If $y \in i[X]^2$, then $y \in i[X]^{2a}$, and since $i[X]^{2a} = (i[X]^{1a})^{1b}$,

$$Y_D(y) = \bigcap\{Y_D(w): w \in i[X]^{1a} \text{ and } y \leq w\} \text{ (6.3 (ii))}.$$

(d) If $z \in i[X]^2$, then $z \in i[X]^{2b}$, so that since $i[X]^{2b} = (i[X]^{1b})^{1a}$,

$$Y_I(z) = \bigcap\{Y_I(w): w \in i[X]^{1b} \text{ and } w \leq z\} \text{ (6.3 (i))}.$$

It now follows from parts (b) and (d) that if $q \in i[X]^2 - (i[X]^{1a} \cup i[X]^{1b})$, then $Y_D(q)$ and $Y_I(q)$ are intersections of closed sets in the topology on Y generated by $\Sigma^{(i)}$, and hence are closed in that topology. Thus $\Sigma^{(i)}$ is a subbase for the interval topology on Y.

To see that $\Sigma^{(i)}$ is a T_1-subbase for Y, let $z \in Y$ and $\mathcal{H} = \{S \in \Sigma^{(i)} : z \in S\}$. Clearly $\mathcal{H}$ is linked. To see that $\mathcal{H}$ is a maximal linked system of $\Sigma^{(i)}$, suppose that $T \in \Sigma^{(i)}$ such that $T \notin \mathcal{H}$. Then $z \notin T$. We will assume that $T \in \Sigma_D^{(i)}$ and note that a dual argument should be used when $T \in \Sigma_I^{(i)}$. Thus there is some $y \in (i[X]^{1a} - i[X]^1) \cup i[X]$ such that $T = Y_D(y)$ and $z \not\leq y$. We consider the possibilities for z.

(e) $z \in (i[X]^{1b} - i[X]^1) \cup i[X]$.

(f) $z \in i[X]^{1a}$.

(g) $z \in i[X]^2 - (i[X]^{1a} \cup i[X]^{1b})$.

(e) If $z \in (i[X]^{1b} - i[X]^1) \cup i[X]$, then $S = Y_I(z) \in \Sigma^{(i)}$, $z \in S$, and $S \cap Y_D(y) = Y_I(z) \cap Y_D(y) = \emptyset$.

(f) If $z \in i[X]^{1a}$, then since $z \not\leq y$, it is possible to pick some $w \in i[X]$ such that $w \leq z$ and $w \not\leq y$. Thus $S = Y_I(w) \in \Sigma^{(i)}$, $z \in S$, and $S \cap Y_D(y) = Y_I(w) \cap Y_D(y) = \emptyset$.

(g) If $z \in i[X]^2 - (i[X]^{1a} \cup i[X]^{1b})$, then since $z \in i[X]^{2b}$ and $z \not\leq y$, there is some $q \in i[X]^{1b}$ such that $q \leq z$ and $q \not\leq y$. If $q \notin i[X]^1$, then $S = Y_I(q) \in \Sigma^{(i)}$, $z \in S$, and $S \cap Y_D(y) = Y_I(q) \cap Y_D(y) = \emptyset$. If $q \in i[X]^1 \subseteq i[X]^{1a}$, then since $q \not\leq y$, there is some $w \in i[X]$ such that $w \leq q$ and $w \not\leq y$. Hence $S = Y_I(w) \in \Sigma_I^{(i)}$, $z \in S$ since $w \leq q \leq z$, and $S \cap Y_D(y) = Y_I(w) \cap Y_D(y) = \emptyset$.

It follows from the arguments in (e), (f), and (g) that in all cases there is some $S \in \Sigma_I^{(i)}$ such that $z \in S$ and $S \cap T = \emptyset$. Clearly $S \in \mathcal{H}$ so that $\mathcal{H} \cup \{T\}$ is not linked. Thus $\mathcal{H}$ is a maximal linked system of $\Sigma^{(i)}$. By Proposition 2.13, $\Sigma^{(i)}$ is a T_1-subbase for Y; therefore, $\Sigma^{(i)}$ is an admissible subbase for Y (4.2 (i)).

(ii) Let $\Sigma = \{i^{-1}[T]: T \in \Sigma^{(i)}\}$. Since i is continuous, it is clear that Σ is a collection of closed subsets of X. Also, since i is a lattice inclusion, it follows that $\{i^{-1}[T]: T \in \Sigma_D^{(i)}\}$ is a collection of ideals of X and $\{i^{-1}[T]: T \in \Sigma_I^{(i)}\}$ is a collection of dual ideals of X. Thus $\Sigma \subseteq \Gamma$. It follows from the definition of $\Sigma^{(i)}$ (6.15) that Σ contains all the principal ideals and principal dual ideals of X. Hence by Proposition 4.3, Σ is an admissible subbase for X.

In view of Proposition 6.17, the subbase $\Sigma^{(i)}$ which we have chosen for Y (where Y is a topological lattice completion of X with respect to i) satisfies the first two conditions in Remark 6.14. It is not true, however, that $\Sigma^{(i)}$ necessarily satisfies the third condition; i.e., it is not always true that if $S, T \in \Sigma^{(i)}$ such that $S \cap T \neq \emptyset$, then $i^{-1}[S] \cap i^{-1}[T] \neq \emptyset$. An example of this can be seen in 8.8. Thus we are lead to define the notion of "lattice density."

DEFINITION 6.18. If Y is a topological lattice completion of X with respect to $i: X \to Y$, then we say X is lattice dense in Y whenever the following condition is satisfied:

If $y \in i[X]^{1b} - i[X]^1$ and $z \in i[X]^{1a} - i[X]^1$ with the property that $y < z$, then there is some $x \in X$ such that $y < i(x) < z$.

It should be noted that if $Y = i[X]^1$, then X is lattice dense in Y since $i[X]^{1a} - i[X]^1 = i[X]^{1b} - i[X]^1 = \emptyset$.

PROPOSITION 6.19. Suppose that Y is a topological lattice completion of X with respect to $i: X \to Y$ and X is lattice dense in Y. If $S, T \in \Sigma^{(i)}$ such that $S \cap T \neq \emptyset$, then $i^{-1}[S] \cap i^{-1}[T] \neq \emptyset$.

PROOF. If either $T = Y$ or $S = Y$, then since $i^{-1}[Y] = X$, the proof is trivial. If T and S are both in $\Sigma_D^{(i)}$ or both in $\Sigma_I^{(i)}$, then since $\{i^{-1}[R]: R \in \Sigma_D^{(i)}\}$ is a collection of closed ideals of X and dually $\{i^{-1}[R]: R \in \Sigma_I^{(i)}\}$ is a collection of closed dual ideals of X, the proof again is trivial (4.5 (i)). In the remaining case we may assume without loss of generality that there exist $t \in (i[X]^{1a} - i[X]^1) \cup i[X]$ and $s \in (i[X]^{1b} - i[X]^1) \cup i[X]$ such that $Y_D(t) = T$ and $Y_I(s) = S$. Since $S \cap T \neq \emptyset$, it must be the case that $s \leq t$. If $s \in i[X]$, then $s = i(x)$ for some $x \in X$ and $x \in i^{-1}[Y_I(i(x))] \cap i^{-1}[Y_D(t)] = i^{-1}[S] \cap i^{-1}[T]$. Dually, if $t \in i[X]$, then $t = i(x)$ for some $x \in X$ and $x \in i^{-1}[Y_D(t)] \cap i^{-1}[Y_I(s)] = i^{-1}[S] \cap i^{-1}[T]$. Thus suppose that $s, t \notin i[X]$. Since neither s nor t is in $i[X]^1$ and $s \leq t$, it follows

that $s < t$. Now since X is lattice dense in Y, there is some $x \in X$ with $s < i(x) < t$; therefore, $x \in i^{-1}[Y_D(t)] \cap i^{-1}[Y_I(s)]$ $= i^{-1}[S] \cap i^{-1}[T]$.

THEOREM 6.20. Given that Y is a topological lattice completion of X with respect to the inclusion map $i: X \to Y$, X is lattice dense in Y, and $\Sigma = \{i^{-1}[T]: T \in \Sigma^{(i)}\}$, then Y is iseomorphic with $\lambda_\Sigma X$ via an iseomorphism $f: \lambda_\Sigma X \to Y$ with the property that $f \circ e_\Sigma = i$.

PROOF. By the definition of topological lattice completions (6.10), i is a continuous lattice inclusion. Σ and $\Sigma^{(i)}$ are admissible subbases for X and Y, respectively, such that $\{i^{-1}[S]: S \in \Sigma_D^{(i)}\} = \Sigma_D$ and $\{i^{-1}[S]: S \in \Sigma_I^{(i)}\} = \Sigma_I$ (6.17). If $S, T \in \Sigma^{(i)}$ such that $S \cap T \neq \emptyset$, then $i^{-1}[S] \cap i^{-1}[T] \neq \emptyset$ (6.19). Thus X, Y, i, Σ, and $\Sigma^{(i)}$ satisfy the hypotheses of Theorem 5.11. Since Y is complete, there is an iseomorphism $f: \lambda_\Sigma X \to Y$ with the property that $f \circ e_\Sigma = i$ (5.12).

We now consider the converse question; i.e., which lattice superextensions are topological lattice completions of X in which X is lattice dense?

PROPOSITION 6.21. If Σ is an admissible subbase for X, then for each $T \in \Sigma_D$ (dually, $T \in \Sigma_I$), $\vee T^+ \in e_\Sigma[X]^{la}$ (dually, $\wedge T^+ \in e_\Sigma[X]^{lb}$).

PROOF. Suppose that $T \in \Sigma_D$. Thus $\vee T^+ = \vee e_\Sigma[T]$ (4.10 (iv)). It follows that $\vee T^+ = \vee \{e_\Sigma(x): e_\Sigma(x) \in e_\Sigma[X] \text{ and } e_\Sigma(x) \leq \vee T^+\}$; therefore, $\vee T^+ \in e_\Sigma[X]^{1a}$. A dual argument shows that if $T \in \Sigma_I$, then $\wedge T^+ \in e_\Sigma[X]^{1b}$.

THEOREM 6.22. *If Σ is an admissible subbase for X, then $\lambda_\Sigma X$ is a topological lattice completion of X with respect to $e_\Sigma: X \to \lambda_\Sigma X$.*

PROOF. Let Σ be an admissible subbase for X. Recall that e_Σ is a continuous lattice inclusion (2.16 (v) and 4.9) and that for each $\mathcal{L} \in \lambda_\Sigma X$,

$$\vee \{\wedge S^+: S \in \mathcal{L} \cap \Sigma_I\} = \mathcal{L} = \wedge \{\vee S^+: S \in \mathcal{L} \cap \Sigma_D\} \text{ (4.10 (vii))}.$$

Thus since for each $S \in \mathcal{L} \cap \Sigma_I$, $\wedge S^+ \leq \mathcal{L}$ (4.10 (vi)) and $\wedge S^+ \in e_\Sigma[X]^{1b}$ (6.21), it follows that

$$\mathcal{L} = \vee \{\mathcal{G} \in \lambda_\Sigma X: \mathcal{G} \in e_\Sigma[X]^{1b} \text{ and } \mathcal{G} \leq \mathcal{L}\} \in (e_\Sigma[X]^{1b})^{1a} = e_\Sigma[X]^{2b}.$$

Dually, since for each $S \in \mathcal{L} \cap \Sigma_D$, $\vee S^+ \in e_\Sigma[X]^{1a}$ (6.21) and $\mathcal{L} \leq \vee S^+$ (4.10 (vi)), then

$$\mathcal{L} = \wedge \{\mathcal{G} \in \lambda_\Sigma X: \mathcal{G} \in e_\Sigma[X]^{1a} \text{ and } \mathcal{L} \leq \mathcal{G}\} \in (e_\Sigma[X]^{1a})^{1b} = e_\Sigma[X]^{2a}.$$

Thus for each $\mathcal{L} \in \lambda_\Sigma X$, $\mathcal{L} \in e_\Sigma[X]^{2a} \cap e_\Sigma[X]^{2b} = e_\Sigma[X]^2$; i.e., $\lambda_\Sigma X = e_\Sigma[X]^2$ and so $\lambda_\Sigma X$ is a topological lattice completion of X with respect to $e_\Sigma: X \to \lambda_\Sigma X$.

We conclude this chapter by exhibiting those lattice superextensions of X in which X is lattice dense.

DEFINITION 6.23. Suppose that Σ is an admissible subbase for X. Then for each $\mathcal{L} \in \lambda_\Sigma X$, we let

$X(D,\mathcal{L}) = \{x \in X: e_\Sigma(x) \leq \mathcal{L}\}$ and

$X(I,\mathcal{L}) = \{x \in X: \mathcal{L} \leq e_\Sigma(x)\}$.

REMARK 6.24. It is easily seen from Definition 6.23 that for each $\mathcal{L} \in \lambda_\Sigma X$,

$$\vee e_\Sigma[X(D,\mathcal{L})] \leq \mathcal{L} \leq \wedge e_\Sigma[X(I,\mathcal{L})].$$

Hence exactly one of the following statements is valid:

(i) $\vee e_\Sigma[X(D,\mathcal{L})] = \mathcal{L} = \wedge e_\Sigma[X(I,\mathcal{L})]$

(ii) $\vee e_\Sigma[X(D,\mathcal{L})] = \mathcal{L} < \wedge e_\Sigma[X(I,\mathcal{L})]$

(iii) $\vee e_\Sigma[X(D,\mathcal{L})] < \mathcal{L} = \wedge e_\Sigma[X(I,\mathcal{L})]$

(iv) $\vee e_\Sigma[X(D,\mathcal{L})] < \mathcal{L} < \wedge e_\Sigma[X(I,\mathcal{L})]$.

Clearly for each $\mathcal{L} \in \lambda_\Sigma X$, $X(D,\mathcal{L})$ is a closed ideal of X which may or may not be in Σ and $X(I,\mathcal{L})$ is a closed dual ideal of X which may or may not be in Σ.

DEFINITION 6.25. If Σ is an admissible subbase for X, then Σ will be called <u>thick</u> if and only if for each pair $\mathcal{L}, \mathcal{M} \in \lambda_\Sigma X$ such that $\mathcal{L}$ satisfies condition (ii) in Remark 6.24 and $\mathcal{M}$ satisfies condition (iii) in Remark 6.24, it follows that $\Theta = \Sigma \cup \{X(D,\mathcal{L}), X(I,\mathcal{M})\}$ is an admissible subbase for X and there exists an iseomorphism $f: \lambda_\Sigma X \to \lambda_\Theta X$ such that $f \circ e_\Sigma = e_\Theta$.

THEOREM 6.26. *If Σ is a thick, admissible subbase for X, then X is lattice dense in $\lambda_\Sigma X$.*

PROOF. To see that X is lattice dense in $\lambda_\Sigma X$, let $\mathcal{L}, \mathcal{M} \in \lambda_\Sigma X$ such that $\mathcal{L} \in e_\Sigma[X]^{1a} - e_\Sigma[X]^1$, $\mathcal{M} \in e_\Sigma[X]^{1b} - e_\Sigma[X]^1$, and $\mathcal{M} < \mathcal{L}$. Thus

$$\mathcal{L} = \vee \{e_\Sigma(x): x \in X \text{ and } e_\Sigma(x) \leq \mathcal{L}\} = \vee e_\Sigma[X(D,\mathcal{L})] \text{ (6.23)}.$$

Since $\mathcal{L} \notin e_\Sigma[X]^1$, it must be the case that

$$\mathcal{L} < \wedge \{e_\Sigma(x): x \in X \text{ and } \mathcal{L} \leq e_\Sigma(x)\} = \wedge e_\Sigma[X(I,\mathcal{L})] \text{ (6.23)}.$$

Hence $\mathcal{L}$ satisfies condition (ii) of Remark 6.24. By a dual argument we see that $\mathcal{M}$ satisfies condition (iii) of Remark 6.24. Thus since Σ is thick, $\lambda_\Sigma X$ is iseomorphic with $\lambda_\Theta X$ where $\Theta = \Sigma \cup \{X(D,\mathcal{L}), X(I,\mathcal{M})\}$ (6.25).

Let $f: \lambda_\Sigma X \to \lambda_\Theta X$ denote the iseomorphism. Since f is a lattice isomorphism, $f \circ e_\Sigma = e_\Theta$ (6.25), and $X(D,\mathcal{L}) \in \Theta_D$, it follows that

$$f(\mathcal{L}) = f(\vee e_\Sigma[X(D,\mathcal{L})]) = \vee f[e_\Sigma[X(D,\mathcal{L})]] \text{ (2.2)}$$
$$= \vee e_\Theta[X(D,\mathcal{L})] = \vee X(D,\mathcal{L})^+ \text{ (4.10 (iv))}.$$

Dually, since $X(I,\mathcal{M}) \in \Theta_I$,

$$f(\mathcal{M}) = \wedge X(I,\mathcal{M})^+.$$

Thus $X(D,\mathcal{L}) \in f(\mathcal{L})$ and $X(I,\mathcal{M}) \in f(\mathcal{M})$ (4.10 (vi)). Since f preserves the order of $\lambda_\Sigma X$ (2.5) and $\mathcal{M} < \mathcal{L}$, it follows that $f(\mathcal{M}) \leq f(\mathcal{L})$ and so

$$X(I,\mathcal{M}) \in f(\mathcal{M}) \cap \Theta_I \subseteq f(\mathcal{L}) \cap \Theta_I \text{ (4.7)};$$

therefore $X(D,\mathcal{L}), X(I,\mathcal{M}) \in f(\mathcal{L})$ and so $X(D,\mathcal{L}) \cap X(I,\mathcal{M}) \neq \emptyset$.

Let $x \in X(D,\mathcal{L}) \cap X(I,\mathcal{M})$. Then

$$f(\mathcal{M}) \leq e_\Theta(x) \leq f(\mathcal{L}).$$

Since $f \circ e_\Sigma = e_\Theta$, we have $f(\mathcal{M}) \leq f(e_\Sigma(x)) \leq f(\mathcal{L})$, and since f is a lattice isomorphism, $\mathcal{M} \leq e_\Sigma(x) \leq \mathcal{L}$. Finally, since $\mathcal{M} \notin e_\Sigma[X]$ and $\mathcal{L} \notin e_\Sigma[X]$, it follows that $\mathcal{M} < e_\Sigma(x) < \mathcal{L}$. Consequently, X is lattice dense in $\lambda_\Sigma X$.

PROPOSITION 6.27. *If Y is a topological lattice completion of X with respect to $i: X \to Y$ and X is lattice dense in Y, then $\Sigma = \{i^{-1}[S]: S \in \Sigma^{(i)}\}$ is a thick, admissible subbase for X.*

PROOF. Recall that Σ is an admissible subbase for X (6.17 (ii)). By Theorem 6.22, Y is iseomorphic with $\lambda_\Sigma X$. To see that Σ is a thick subbase for X, suppose that $\mathcal{L}, \mathcal{M} \in \lambda_\Sigma X$ such that $\mathcal{L}$ satisfies condition (ii) of Remark 6.24 and $\mathcal{M}$ satisfies condition (iii) of Remark 6.24. Since

$$\vee \{e_\Sigma(x): x \in X \text{ and } e_\Sigma(x) \leq \mathcal{L}\}$$
$$= \mathcal{L} < \wedge \{e_\Sigma(x): x \in X \text{ and } \mathcal{L} \leq e_\Sigma(x)\} \quad (6.23, 6.24),$$

it follows that $\mathcal{L} \in e_\Sigma[X]^{1a} - e_\Sigma[X]^1$. Thus $\{\mathcal{G} \in \lambda_\Sigma X: \mathcal{G} \leq \mathcal{L}\} \in \Sigma_D^{(i)}$ (6.15), and so $X(D,\mathcal{L}) = i^{-1}[\{\mathcal{G} \in \lambda_\Sigma X: \mathcal{G} \leq \mathcal{L}\}] \in \Sigma_D$. Dually, $\mathcal{M} \in e_\Sigma[X]^{1b} - e_\Sigma[X]^1$, $\{\mathcal{G} \in \lambda_\Sigma X: \mathcal{M} \leq \mathcal{G}\} \in \Sigma_I^{(i)}$, and so $X(I,\mathcal{M}) = i^{-1}[\{\mathcal{G} \in \lambda_\Sigma X: \mathcal{M} \leq \mathcal{G}\}] \in \Sigma_I$. Consequently, $\Theta = \Sigma \cup \{X(D,\mathcal{L}), X(I,\mathcal{M})\} = \Sigma$ is an admissible subbase for X, and there is an iseomorphism $f: \lambda_\Sigma X \to \lambda_\Theta X$, namely, the identity map, such that $f \circ e_\Sigma = e_\Theta$. Hence Σ is thick.

PROPOSITION 6.28. *The usual subbase T and the subbase Γ are both thick.*

PROOF. To see that T is thick, we recall that $\lambda_T X$ is the completion by cuts of X so that $\lambda_T X = e_T[X]^1$. Thus it follows that every member of $\lambda_T X$ satisfies condition (i) in Remark 6.24. Hence T vacuously satisfies the definition of a thick subbase.

To see that Γ is thick, suppose that $\mathcal{L}$, $\mathcal{M} \in \lambda_\Gamma X$ such that $\mathcal{L}$ satisfies condition (ii) of Remark 6.24 and $\mathcal{M}$ satisfies condition (iii) of Remark 6.24. Since $X(D,\mathcal{L})$ is a closed ideal of X and $X(I,\mathcal{M})$ is a closed dual of X, then $X(D,\mathcal{L}) \in \Gamma_D$ and $X(I,\mathcal{M}) \in \Gamma_I$ (4.1 (ii)). Thus $\Theta = \Gamma \cup \{X(D,\mathcal{L}), X(I,\mathcal{M})\} = \Gamma$ is an admissible subbase for X (4.4) and the identity map $f: \lambda_\Gamma X \to \lambda_\Theta X$ is an iseomorphism such that $f \circ e_\Gamma = e_\Theta$.

COROLLARY 6.29. X is lattice dense in $\lambda_T X$ and $\lambda_\Gamma X$.

PROOF. Since both T and Γ are thick (6.28), admissible subbases for X (4.4), the proof is immediate from Theorem 6.26.

All of the admissible subbases which are considered in the examples (Chapter 8) are thick subbases for X. Whether or not every admissible subbase for X is thick remains an open question.

THEOREM 6.30. Suppose that X is a topological lattice and Θ is an admissible subbase for X. Then X is lattice dense in $\lambda_\Theta X$ if and only if there exists a thick admissible subbase Σ for X such that $\lambda_\Theta X$ is iseomorphic with $\lambda_\Sigma X$ via an iseomorphism $f: \lambda_\Sigma X \to \lambda_\Theta X$ with the property that $f \circ e_\Sigma = e_\Theta$.

PROOF. Let Θ be an admissible subbase for X. Thus $\lambda_\Theta X$ is a topological lattice completion of X with respect to $e_\Theta: X \to \lambda_\Theta X$ (6.22).

If X is lattice dense in $\lambda_\Theta X$, then $\Sigma = \{e_\Theta^{-1}[S]: S \in \Sigma^{(e_\Theta)}\}$ is a thick admissible subbase for X (6.27) and $\lambda_\Theta X$ is iseomorphic with $\lambda_\Sigma X$ via an iseomorphism $f: \lambda_\Sigma X \to \lambda_\Theta X$ with the property that $f \circ e_\Sigma = e_\Theta$ (6.20).

Conversely, if there is some thick admissible subbase Σ for X and an iseomorphism $f: \lambda_\Sigma X \to \lambda_\Theta X$ such that $f \circ e_\Sigma = e_\Theta$, then since X is lattice dense in $\lambda_\Sigma X$ (6.26), it follows that X is lattice dense in $\lambda_\Theta X$.

7. LARGEST AND SMALLEST LATTICE SUPEREXTENSIONS

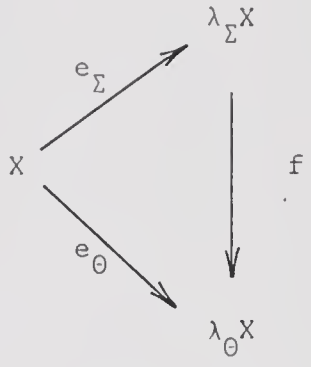
In this chapter we will consider the possible existence of certain maps between different lattice superextensions of the same topological lattice. The results will then be used to establish the existence of maximal and minimal lattice superextensions for all topological lattices and in certain cases, largest and smallest lattice superextenions (Theorem 7.15). Thus in the realm of lattice superextensions we have analogues to the Stone-Čech and Alexandroff compactifications. Kaufman [12] obtained similar analogues for the totally ordered case. Theorem 7.15 is a generalization of Kaufman's results.

In this chapter, as in the previous chapters, X will denote a topological lattice. Also, to avoid confusion, we will sometimes use the symbol "*" in place of "+" to denote the subbasic closed sets of a lattice superextension; e.g., if Θ is an admissible subbase for X, then for each $T \in \Theta$,

$$T^* = \{\mathcal{L} \in \lambda_\Theta X: T \in \mathcal{L}\} \text{ and } \Theta^* = \{T^*: T \in \Theta\}.$$

DEFINITION 7.1. Let Θ and Σ be admissible subbases for X.

(i) We say that $\lambda_\Sigma X$ algebraically dominates $\lambda_\Theta X$ if and only if there is a surjective, order-preserving function $f: \lambda_\Sigma X \to \lambda_\Theta X$ such that the diagram



commutes (i.e., $f \circ e_\Sigma = e_\Theta$) and such that f preserves the suprema of ideals in X and the infima of dual ideals in X; i.e., if I is an ideal of X and D is a dual ideal of X, then $f(\vee e_\Sigma[I]) = \vee (f \circ e_\Sigma)[I]$ and $f(\wedge e_\Sigma[D]) = \wedge (f \circ e_\Sigma)[D]$. Such a function is called a dominating function of $\lambda_\Sigma X$ over $\lambda_\Theta X$.

(ii) We say that the superextension $\lambda_\Sigma X$ dominates the superextension $\lambda_\Theta X$ if and only if $\lambda_\Sigma X$ algebraically dominates $\lambda_\Theta X$ via a continuous dominating function.

It might be noted that although $\lambda_\Sigma X$ and $\lambda_\Theta X$ are both lattices, a dominating function f is not required to preserve all suprema and infima of $\lambda_\Sigma X$; i.e., f is not required to be a lattice homomorphism. The reason for this is apparent from Example 8.9 in which a topological lattice X is exhibited for which the lattice $\lambda_\Gamma X$ algebraically dominates the lattice $\lambda_T X$ (and also for which the superextension $\lambda_\Gamma X$ dominates

the superextension $\lambda_T X$), but for which there are $\mathcal{L}, \mathcal{M} \in \lambda_\Gamma X$ such that $f(\mathcal{L}) \wedge f(\mathcal{M}) \neq f(\mathcal{L} \wedge \mathcal{M})$. (See 8.9 for details.) Example 8.7 shows that algebraic domination does not necessarily imply superextension domination, for in this example there exists a unique algebraic dominating function $f: \lambda_\Gamma X \to \lambda_T X$ which is not continuous.

PROPOSITION 7.2. *Suppose that Θ and Σ are admissible subbases for X such that $\lambda_\Sigma X$ algebraically dominates $\lambda_\Theta X$ via a dominating function $f: \lambda_\Sigma X \to \lambda_\Theta X$. Then*

(i) *for each $R \in \Sigma_D$ (dually, $R \in \Sigma_I$), $f(\vee R^+) = \vee e_\Theta[R]$ (dually, $f(\wedge R^+) = \wedge e_\Theta[R]$),*

(ii) *for each $R \in \Sigma_D \cap \Theta_D$ (dually, $R \in \Sigma_I \cap \Theta_I$), $f(\vee R^+) = \vee R^*$ (dually, $f(\wedge R^+) = \wedge R^*$), and*

(iii) *for each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta) \subseteq f(\mathcal{L})$.*

PROOF. (i) Suppose that $R \in \Sigma_D$. Then by the hypotheses on f and since R is an ideal of X, it follows that

$$f(\vee R^+) = f(\vee e_\Sigma[R]) \quad (4.10\ (iv))$$
$$= \vee (f \circ e_\Sigma)[R] = \vee e_\Theta[R].$$

If $R \in \Sigma_I$, then a dual argument shows that $f(\wedge R^+) = \wedge e_\Theta[R]$.

(ii) Suppose that $R \in \Sigma_D \cap \Theta_D$. Then since $f(\vee R^+) = \vee e_\Theta[R]$ ((i) above) and $\vee e_\Theta[R] = \vee R^*$ (4.10 (iv)), we see that $f(\vee R^+) = \vee R^*$. Dually, if $R \in \Sigma_I \cap \Theta_I$, then $f(\wedge R^+) = \wedge R^*$.

(iii) Suppose that $\mathcal{L} \in \lambda_\Sigma X$. Let $R \in (\mathcal{L} \cap \Theta_D)$. Thus $\mathcal{L} \leq \vee R^+$ (4.10 (vi)), and since f is order-preserving, $f(\mathcal{L}) \leq f(\vee R^+) = \vee R^*$ ((ii) above); i.e., $R \in f(\mathcal{L})$ (4.10 (vi)). Hence $(\mathcal{L} \cap \Theta_D) \leq f(\mathcal{L})$, and by a dual argument, $(\mathcal{L} \cap \Theta_I) \subseteq f(\mathcal{L})$. Therefore, $(\mathcal{L} \cap \Theta) \subseteq f(\mathcal{L})$.

LEMMA 7.3. *If Σ is an admissible subbase for X, then for each $a \in X$, $\vee e_\Sigma[X_D(a)] = e_\Sigma(a)$ and $\wedge e_\Sigma[X_I(a)] = e_\Sigma(a)$.*

PROOF. Let $a \in X$. Then for each $x \in X_D(a)$, $x \leq a$ so that $e_\Sigma(x) \leq e_\Sigma(a)$ since e_Σ is a lattice inclusion (4.9); therefore, $\vee e_\Sigma[X_D(a)] \leq e_\Sigma(a)$. Since $a \in X_D(a)$, it follows that $e_\Sigma(a) = \vee e_\Sigma[X_D(a)]$.

A dual argument shows that $e_\Sigma(a) = \wedge e_\Sigma[X_I(a)]$.

PROPOSITION 7.4. *If Θ and Σ are admissible subbases for X such that $\Theta \subseteq \Sigma$, then $\lambda_\Sigma X$ algebraically dominates $\lambda_\Theta X$.*

PROOF. We define $f: \lambda_\Sigma X \to \lambda_\Theta X$ by

$f(\mathcal{L}) = \vee \{\wedge e_\Theta[S]: S \in (\mathcal{L} \cap \Sigma_I)\}$ for each $\mathcal{L} \in \lambda_\Sigma X$.

$f \circ e_\Sigma = e_\Theta$: Let $x \in X$. For each $S \in e_\Sigma(x) \cap \Sigma_I$, $x \in S$ so that $\wedge e_\Theta[S] \leq e_\Theta(x)$; therefore, $f(e_\Sigma(x)) = \vee \{\wedge e_\Theta[S]: S \in e_\Sigma(x) \cap \Sigma_I\} \leq e_\Theta(x)$. Now suppose that $T \in e_\Theta(x) \cap \Theta_I$. Thus $x \in T$ and since $T \in \Theta_I \subseteq \Sigma_I$, it follows that $T \in e_\Sigma(x)$. Hence by Proposition 4.10 (iv) and the definition of f, $\wedge T^* = \wedge e_\Theta[T] \leq f(e_\Sigma(x))$; i.e., $T \in f(e_\Sigma(x)) \cap \Theta_I$ (4.10 (vi)) and so $e_\Theta(x) \leq f(e_\Sigma(x))$ (4.7). It follows that $f \circ e_\Sigma(x) = e_\Theta(x)$ for each $x \in X$.

f preserves the order of $\lambda_\Sigma X$: Suppose that $\mathcal{L}, \mathcal{M} \in \lambda_\Sigma X$ such that $\mathcal{L} \leq \mathcal{M}$. Thus $(\mathcal{L} \cap \Sigma_I) \subseteq (\mathcal{M} \cap \Sigma_I)$ (4.7) and so $f(\mathcal{L}) = \vee \{\wedge e_\Theta[S]: S \in \mathcal{L} \cap \Sigma_I\} \leq \vee \{\wedge e_\Theta[S]: S \in \mathcal{M} \cap \Sigma_I\} = f(\mathcal{M})$.

For each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta) \subseteq f(\mathcal{L})$: Let $\mathcal{L} \in \lambda_\Sigma X$. If $T \in \mathcal{L} \cap \Theta_I$, then by the definition of f, $\wedge e_\Theta[T] \leq f(\mathcal{L})$. Since $\wedge e_\Theta[T] = \wedge T^*$ (4.10 (iv)), it follows that $T \in f(\mathcal{L})$ (4.10 (vi)), and so $(\mathcal{L} \cap \Theta_I) \subseteq f(\mathcal{L})$. If $T \in (\mathcal{L} \cap \Theta_D)$, then for each $S \in (\mathcal{L} \cap \Theta_I)$, $T \cap S \neq \emptyset$. Thus there is some $x \in S \cap T$ and so $\wedge e_\Theta[S] \leq e_\Theta(x) \leq \vee e_\Theta[T]$. It follows that

$$f(\mathcal{L}) = \vee \{\wedge e_\Theta[S]: S \in (\mathcal{L} \cap \Sigma_I)\} \leq \vee e_\Theta[T] = \vee T^* \text{ (4.10 (iv))};$$

therefore, $T \in f(\mathcal{L})$ (4.10 (vi)) and so $(\mathcal{L} \cap \Theta_D) \subseteq f(\mathcal{L})$.

f is surjective: If $\mathcal{M} \in \lambda_\Theta X$, then $\mathcal{M}$ is a linked system of Σ and hence is contained in some maximal linked system $\mathcal{L}$ of Σ. Since $\mathcal{M} \subseteq (\mathcal{L} \cap \Theta) \subseteq f(\mathcal{L})$, it follows from the maximality of $\mathcal{M}$ that $f(\mathcal{L}) = \mathcal{M}$.

f preserves suprema of ideals and infima of dual ideals of X: Suppose that I is an ideal of X and $\mathcal{L} = \vee e_\Sigma[I]$. Since f preserves the order of $\lambda_\Sigma X$, we see that for each $x \in I$, $f(e_\Sigma(x)) = e_\Theta(x) \leq f(\mathcal{L})$ and so $\vee e_\Theta[I] \leq f(\mathcal{L})$. If $T \in (\vee e_\Theta[I] \cap \Theta_D)$, then for each $x \in I$, $T \in e_\Theta(x)$ (4.8) and so $x \in T$; consequently, $I \subseteq T$. Since $T \in \Sigma_D$, it follows that $T \in \vee e_\Sigma[I] = \mathcal{L}$ (4.8). Thus $T \in (\mathcal{L} \cap \Theta) \subseteq f(\mathcal{L})$ and so $\vee e_\Theta[I] \cap \Theta_D \subseteq f(\mathcal{L})$. By the definition of the order on $\lambda_\Theta X$ (4.7), $f(\mathcal{L}) \leq \vee e_\Theta[I]$; therefore $f(\mathcal{L}) = f(\vee e_\Sigma[I]) = \vee e_\Theta[I] = \vee (f \circ e_\Sigma)[I]$. A dual argument shows that f preserves the infima of dual ideals of X.

In view of the above arguments, $\lambda_\Sigma X$ algebraically dominates $\lambda_\Theta X$.

PROPOSITION 7.5. *Suppose that Θ and Σ are admissible subbases for X such that $\Theta \subseteq \Sigma$.*

(i) *If for each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta)$ is contained in a unique maximal linked system of Θ, then there is a unique dominating function $f: \lambda_\Sigma X \to \lambda_\Theta X$. Moreover, f may be defined by $f(\mathcal{L}) = \{T \in \Theta: (\mathcal{L} \cap \Theta) \cup \{T\}$ is linked$\}$ for each $\mathcal{L} \in \lambda_\Sigma X$.*

(ii) *If for each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta)$ is a maximal linked system of Θ, then there is a unique and continuous dominating function $f: \lambda_\Sigma X \to \lambda_\Theta X$.*

PROOF. (i) Since $\Theta \subseteq \Sigma$, there is a dominating function $f: \lambda_\Sigma X \to \lambda_\Theta X$ (7.4). Suppose that $g: \lambda_\Sigma X \to \lambda_\Theta X$ is also a dominating function. Thus for each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta) \subseteq g(\mathcal{L})$ and $(\mathcal{L} \cap \Theta) \subseteq f(\mathcal{L})$ (7.2 (iii)). Since $(\mathcal{L} \cap \Theta)$ is contained in a unique maximal linked system of Θ, it follows that $g(\mathcal{L}) = f(\mathcal{L})$ for each $\mathcal{L} \in \lambda_\Sigma X$; i.e., $f = g$ and f is unique.

To see that f can be defined as stated, let $\mathcal{L} \in \lambda_\Sigma X$ and $\mathcal{H} = \{T \in \Theta: (\mathcal{L} \cap \Theta) \cup \{T\}$ is linked$\}$. To see that $\mathcal{H}$ is linked, let $S, T \in \mathcal{H}$. Then $(\mathcal{L} \cap \Theta) \cup \{S\}$ is linked and $(\mathcal{L} \cap \Theta) \cup \{T\}$ is linked. Since $(\mathcal{L} \cap \Theta)$ is contained in a unique maximal linked system $\mathcal{J}$ of Θ, it follows that $(\mathcal{L} \cap \Theta) \cup \{S\}$ and $(\mathcal{L} \cap \Theta) \cup \{T\}$ are both subsets of $\mathcal{J}$. Thus $S \cap T \neq \emptyset$ and $\mathcal{H}$ is linked. To see that $\mathcal{H}$ is a maximal linked system of Θ, suppose that $T \in \Theta$ such that $\mathcal{H} \cup \{T\}$ is linked. Thus $(\mathcal{L} \cap \Theta) \cup \{T\}$ is linked, and so by the definition of $\mathcal{H}$, $T \in \mathcal{H}$.

Since both $\mathcal{H}$ and $f(\mathcal{L})$ are maximal linked systems of Θ which contain $(\mathcal{L} \cap \Theta)$ and $(\mathcal{L} \cap \Theta)$ is contained in a unique maximal linked system of Θ, it follows that $f(\mathcal{L}) = \mathcal{H}$.

(ii) If for each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta)$ is a maximal linked system of Θ, then $(\mathcal{L} \cap \Theta)$ is contained in a unique maximal linked system of Θ. By (i) above, there is a unique dominating function $f: \lambda_\Sigma X \to \lambda_\Theta X$ defined by $f(\mathcal{L}) = \{T \in \Theta: (\mathcal{L} \cap \Theta) \cup \{T\} \text{ is linked}\}$ for each $\mathcal{L} \in \lambda_\Sigma X$. Since $(\mathcal{L} \cap \Theta)$ is a maximal linked system of Θ, it follows that $f(\mathcal{L}) = (\mathcal{L} \cap \Theta)$. Thus for each $T \in \Theta$,

$$f^{-1}[T^*] = \{\mathcal{L} \in \lambda_\Sigma X: f(\mathcal{L}) \in T^*\} = \{\mathcal{L} \in \lambda_\Sigma X: T \in f(\mathcal{L})\}$$
$$= \{\mathcal{L} \in \lambda_\Sigma X: T \in \mathcal{L}\} = T^+ \in \Sigma^+.$$

Since Θ^* is a subbase for the closed sets of $\lambda_\Theta X$, it follows that f is continuous.

DEFINITION 7.6. If Θ and Σ are admissible subbases for X, then we say that Θ is equivalent to Σ whenever $\lambda_\Theta X$ is iseomorphic with $\lambda_\Sigma X$ via an iseomorphism $f: \lambda_\Sigma X \to \lambda_\Theta X$ such that $f \circ e_\Sigma = e_\Theta$.

PROPOSITION 7.7. If Θ and Σ are admissible subbases for X such that each of $\lambda_\Theta X$ and $\lambda_\Sigma X$ algebraically dominates the other, then Θ and Σ are equivalent. Specifically, we will show that if $f: \lambda_\Sigma X \to \lambda_\Theta X$ and $g: \lambda_\Theta X \to \lambda_\Sigma X$ are dominating functions, then

(i) $g \circ f$ is the identity on $\lambda_\Sigma X$ and $f \circ g$ is the identity on $\lambda_\Theta X$

(ii) f and g are injective

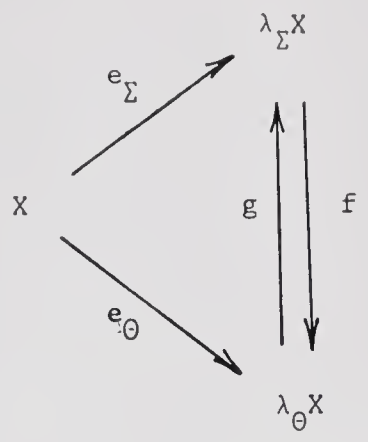
(iii) f and g are closed maps

(iv) f and g are homeomorphisms

(v) f and g are lattice isomorphisms

(vi) $\lambda_\Sigma X$ is iseomorphic with $\lambda_\Theta X$.

PROOF. The hypotheses describe the following diagram:



where $g \circ e_\Theta = e_\Sigma$ and $f \circ e_\Sigma = e_\Theta$.

(i) Let $R \in \Sigma_D$. Thus R is an ideal of X. By the properties of dominating functions (7.1 (i)) and Proposition 4.10 (iv),

$$g \circ f(\vee R^+) = g \circ f(\vee e_\Sigma[R]) = g(\vee f \circ e_\Sigma[R]) =$$

$$g(\vee e_\Theta[R]) = \vee g \circ e_\Theta[R] = \vee e_\Sigma[R] = \vee R^+.$$

Dually, for each $R \in \Sigma_I$, $g \circ f(\wedge R^+) = \wedge R^+$. Now let $\mathcal{L} \in \lambda_\Sigma X$. If $R \in \mathcal{L} \cap \Sigma_D$, then $\mathcal{L} \leq \vee R^+$ (4.10 (vi)), and so $f(\mathcal{L}) \leq f(\vee R^+)$ since f is order-preserving. Thus $g \circ f(\mathcal{L}) \leq g \circ f(\vee R^+) = \vee R^+$ since g is order-preserving. Hence $R \in g \circ f(\mathcal{L})$ (4.10 (vi)); i.e., $\mathcal{L} \cap \Sigma_D \subseteq g \circ f(\mathcal{L})$. Dually, $\mathcal{L} \cap \Sigma_I \subseteq g \circ f(\mathcal{L})$ so that $\mathcal{L} = g \circ f(\mathcal{L})$ and $g \circ f$ is the identity on $\lambda_\Sigma X$.

A similar argument shows that $f \circ g$ is the identity on $\lambda_\Theta X$.

(ii) If $\mathcal{L}$ and $\mathcal{M}$ are elements of $\lambda_\Sigma X$ such that $f(\mathcal{L}) = f(\mathcal{M})$, then by (i) above $\mathcal{L} = g \circ f(\mathcal{L}) = g \circ f(\mathcal{M}) = \mathcal{M}$. Thus f is injective.

A similar argument shows that g is injective.

(iii) Let $R \in \Sigma_D$. If $\mathcal{L} \in R^+$, then $R \in \mathcal{L}$ and so $\mathcal{L} \leq \vee R^+$ (4.10 (vi)). Since f is order-preserving, $f(\mathcal{L}) \leq f(\vee R^+) = \vee e_\Theta[R]$ (7.2 (i)). Thus

$$f[R^+] \subseteq \{\mathcal{J} \in \lambda_\Theta X: \mathcal{J} \leq \vee e_\Theta[R]\}.$$

Conversely, if $\mathcal{J} \in \lambda_\Theta X$ such that $\mathcal{J} \leq \vee e_\Theta[R]$, then since f is surjective, there is some $\mathcal{L} \in \lambda_\Sigma X$ with $f(\mathcal{L}) = \mathcal{J}$. Now since g is order-preserving and $g \circ f$ is the identity on $\lambda_\Sigma X$, it follows that $\mathcal{L} = g \circ f(\mathcal{L}) = g(\mathcal{J}) \leq g(\vee e_\Theta[R])$. Thus by the properties of g (7.1 (i)),

$$\mathcal{L} \leq g(\vee e_\Theta[R]) = \vee (g \circ e_\Theta)[R] = \vee e_\Sigma[R] = \vee R^+ \text{ (4.10 (iv))}.$$

Hence $R \in \mathcal{L}$ (4.10 (vi)) and so $\mathcal{L} \in R^+$: therefore

$$\{\mathcal{J} \in \lambda_\Theta X: \mathcal{J} \leq \vee e_\Theta[R]\} = f[R^+] = g^{-1}[R^+].$$

A dual argument shows that if $R \in \Sigma_I$, then

$$g^{-1}[R^+] = f[R^+] = \{\mathcal{J} \in \lambda_\Theta X: \wedge e_\Theta[R] \leq \mathcal{J}\}.$$

Since Σ^+ is a subbase for the closed sets of $\lambda_\Sigma X$, it follows from the above equalities that $f^{-1} = g$ is continuous. Hence f is a closed map.

A similar argument shows that g is a closed map.

(iv) Since each of f and g is closed, continuous, and bijective, each must be a homeomorphism.

(v) Suppose that C is a non-void subset of $\lambda_\Sigma X$ and $\mathcal{L} = \vee C$. Since f preserves the order of $\lambda_\Sigma X$, it is clear that $\mathcal{M} = \vee \{f(\mathcal{J}): \mathcal{J} \in C\} \leq f(\mathcal{L})$. To see that $f(\mathcal{L}) \leq \mathcal{M}$ let $T \in \mathcal{M} \cap \Theta_D$. Thus $\mathcal{M} \leq \vee T^*$ (4.10 (vi)) so that for each $\mathcal{J} \in C$, $f(\mathcal{J}) \leq \vee T^*$. Since g is order-preserving and $g \circ f$ is the identity on $\lambda_\Sigma X$, it follows that for each $\mathcal{J} \in C$,

$$\mathcal{J} = g \circ f(\mathcal{J}) \leq g(\vee T^*) = \vee e_\Sigma[T] \text{ (7.2 (i))}.$$

Thus $\mathcal{L} = \vee \{\mathcal{G}: \mathcal{G} \in C\} \leq \vee e_\Sigma[T]$. Finally, since f is a dominating function (7.1 (i)), it follows that

$$f(\mathcal{L}) \leq f(\vee e_\Sigma[T]) = \vee f \circ e_\Sigma[T] = \vee e_\Theta[T] = \vee T^* \text{ (4.10 (iv))}.$$

Hence, $T \in f(\mathcal{L})$ (4.10 (vi)). Therefore, $\mathcal{M} \cap \Theta_D \subseteq f(\mathcal{L})$ so that $\mathcal{M} = f(\mathcal{L})$; i.e., $f(\vee C) = \vee f[C]$. A dual argument shows that $f(\wedge C) = \wedge f[C]$ and so since f is bijective, f is a lattice isomorphism.

A similar argument shows that g is a lattice isomorphism.

(vi) That f and g are iseomorphisms follows immediately from (iv) and (v) above.

COROLLARY 7.8. *If Θ and Σ are admissible subbases for X such that $\Theta \subseteq \Sigma$ and $\lambda_\Theta X$ algebraically dominates $\lambda_\Sigma X$, then Θ is equivalent to Σ.*

PROOF. Since $\Theta \subseteq \Sigma$, it follows that $\lambda_\Sigma X$ algebraically dominates $\lambda_\Theta X$ (7.4). By the hypotheses, $\lambda_\Theta X$ algebraically dominates $\lambda_\Sigma X$. Thus Θ is equivalent with Σ (7.7).

PROPOSITION 7.9. *Suppose that Θ is an admissible subbase for X and $\Sigma = \Theta \cup T$. Then*

(i) *Σ is an admissible subbase for X, and for each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta)$ is a maximal linked system of Θ, and*

(ii) *for each $\mathcal{L} \in \lambda_\Sigma X$ and each $a \in X$, if $(\mathcal{L} \cap \Theta) \leq e_\Theta(a)$, then $X_D(a) \in \mathcal{L}$, and dually, if $e_\Theta(a) \leq (\mathcal{L} \cap \Theta)$, then $X_I(a) \in \mathcal{L}$.*

PROOF. (i) Since $T \subseteq \Sigma = \Theta \cup T \subseteq \Gamma$, Σ is an admissible subbase for X (4.3). Let $\mathcal{L} \in \lambda_\Sigma X$. Clearly $(\mathcal{L} \cap \Theta)$ is linked. To see that $(\mathcal{L} \cap \Theta)$ is a maximal linked system of Θ, let $T \in \Theta_D$ such that $(\mathcal{L} \cap \Theta) \cup \{T\}$ is linked. If $T \notin \mathcal{L}$ then there is some $U \in \Sigma_I \cap \mathcal{L}$ such that $T \cap U = \emptyset$. Since $\Sigma_I = \Theta_I \cup T_I$ and $\{T\} \cup (\mathcal{L} \cap \Theta_I)$ is linked, it must be the case that $U \in T_I$; i.e., $U = X_I(u)$ for some $u \in X$. Since $u \notin T$ and Θ is a T_1-subbase for X, there is some $S \in \Theta_I$ such that $u \in S$ and $S \cap T = \emptyset$. It follows that $X_I(u) \subseteq S$, and $S \in (\mathcal{L} \cap \Theta)$; therefore, $(\mathcal{L} \cap \Theta) \cup \{T\}$ is not linked —a contradiction. Thus $T \in \mathcal{L}$. Dually, if $T \in \Theta_I$ such that $(\mathcal{L} \cap \Theta) \cup \{T\}$ is linked, then $T \in \mathcal{L}$, and hence $(\mathcal{L} \cap \Theta)$ is a maximal linked system of Θ.

(ii) By (i) above and Proposition 7.5 (i) there is a unique dominating function $f: \lambda_\Sigma X \to \lambda_\Theta X$ which may be defined by $f(\mathcal{L}) = (\mathcal{L} \cap \Theta)$ for each $\mathcal{L} \in \lambda_\Sigma X$. Suppose that $\mathcal{L} \in \lambda_\Sigma X$ and $a \in X$ such that $(\mathcal{L} \cap \Theta) = f(\mathcal{L}) \leq e_\Theta(a)$. For each $S \in (\mathcal{L} \cap \Sigma_I)$, $\wedge S^+ \leq \mathcal{L}$ (4.10 (vi)) so that $f(\wedge S^+) \leq f(\mathcal{L})$ since f is order-preserving. If $S \in \Theta_I$, then $f(\wedge S^+) = \wedge S^*$ (7.2 (ii)) and since $f(\mathcal{L}) \leq e_\Theta(a)$, it follows that $\wedge S^* \leq e_\Theta(a)$; therefore $a \in S$ so that $S \cap X_D(a) \neq \emptyset$. If $S \notin \Theta_I$, then there is some $b \in X$ such that $S = X_I(b)$. In this case $f(\wedge S^+) = \wedge e_\Theta[S]$ (7.2 (i)) $= \wedge e_\Theta[X_I(b)] = e_\Theta(b)$ (7.3). Thus $e_\Theta(b) \leq f(\mathcal{L}) \leq e_\Theta(a)$; therefore $b \leq a$ since e_Θ is a lattice inclusion (4.9) and $S \cap X_D(a) = X_I(b) \cap X_D(a) \neq \emptyset$. It follows that $\mathcal{L} \cup \{X_D(a)\}$ is linked and so $X_D(a) \in \mathcal{L}$. A dual argument shows the other statement.

PROPOSITION 7.10. If Θ is an admissible subbase for X, then Θ is equivalent to $\Theta \cup T$.

PROOF. Let $\Sigma = \Theta \cup T$. Then Σ is an admissible subbase for X (7.9 (i)). For each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta)$ is a maximal linked system of Θ (7.9 (i)) so that $\lambda_\Sigma X$ algebraically dominates $\lambda_\Theta X$ via a unique, continuous function $f: \lambda_\Sigma X \to \lambda_\Theta X$ which may be defined by $f(\mathcal{L}) = (\mathcal{L} \cap \Theta)$ for each $\mathcal{L} \in \lambda_\Sigma X$ (7.5). To see that f is an iseomorphism, we need only show that f is a closed, injective lattice homomorphism (2.10).

f is injective: Suppose that $\mathcal{L}, \mathcal{M} \in \lambda_\Sigma X$ such that $\mathcal{L} \neq \mathcal{M}$. Then we may assume, without loss of generality, that there exist $T \in (\mathcal{L} \cap \Sigma_I)$ and $S \in (\mathcal{M} \cap \Sigma_D)$ such that $T \cap S = \emptyset$. If both T and S are in Θ, then clearly $f(\mathcal{L}) = (\mathcal{L} \cap \Theta) \neq (\mathcal{M} \cap \Theta) = f(\mathcal{M})$. Suppose that $T \notin \Theta_I$; i.e., there is some $b \in X$ such that $T = X_I(b)$. Note that $e_\Theta(b) \not\leq f(\mathcal{M})$, for if $e_\Theta(b) \leq f(\mathcal{M})$, then $T = X_I(b) \in \mathcal{M}$ (7.9 (ii))--a contradiction. Thus $e_\Theta(b) \not\leq f(\mathcal{M})$ and there is some $T' \in \Theta_I$ such that $T' \in e_\Theta(b)$ and $T' \notin f(\mathcal{M}) = (\mathcal{M} \cap \Theta)$ (4.7). It follows that $X_I(b) \subseteq T'$, $T' \in (\mathcal{L} \cap \Theta) = f(\mathcal{L})$, and hence $f(\mathcal{L}) \neq f(\mathcal{M})$. In the case that $T \in \Theta_I$ and $S \notin \Theta_D$, a dual argument shows that there is some $S' \in \Theta_D$ such that $S \subseteq S'$, $S' \in f(\mathcal{M})$, $S' \notin (\mathcal{L} \cap \Theta) = f(\mathcal{L})$, and hence $f(\mathcal{L}) \neq f(\mathcal{M})$.

f is a lattice homomorphism: Suppose C is a non-void subset of $\lambda_\Sigma X$. Since f is order-preserving, it follows that for each $\mathcal{L} \in C$, $f(\mathcal{L}) \leq f(\vee C)$; therefore, $\vee f[C] \leq f(\vee C)$. If $T \in \vee f[C] \cap \Theta_D$, then

for each $\mathcal{L} \in C$, $T \in f(\mathcal{L}) = (\mathcal{L} \cap \Theta)$ (4.8); hence $T \in \mathcal{L}$ for each $\mathcal{L} \in C$ so that $T \in \vee C$ (4.8). Thus $T \in (\vee C \cap \Theta) = f(\vee C)$; i.e., $(\vee f[C] \cap \Theta_D) \subseteq (f(\vee C) \cap \Theta_D)$. It follows that $f(\vee C) \leq \vee f[C]$, and so $f(\vee C) = \vee f[C]$. A dual argument shows that $f(\wedge C) = \wedge f[C]$.

f is a closed map: Let $R \in \Sigma$. If $R \in \Theta$, then

$$f[R^+] = \{f(\mathcal{L}): R \in \mathcal{L}\} = \{f(\mathcal{L}): R \in (\mathcal{L} \cap \Theta)\}$$
$$= \{f(\mathcal{L}): R \in f(\mathcal{L})\} = \{f(\mathcal{L}): f(\mathcal{L}) \in R^*\}$$
$$= R^* \text{ since } f \text{ is surjective.}$$

If $R \notin \Theta$, then $R \in \Sigma - \Theta \subseteq T$. Suppose that $R \in T_D$; i.e., there is some $a \in X$ such that $R = X_D(a)$. Thus

$$f[R^+] = \{f(\mathcal{L}): \mathcal{L} \leq e_\Sigma(a)\} \qquad (4.10 \text{ (vi), (iv) and } 7.3)$$
$$= \{f(\mathcal{L}): (\mathcal{L} \cap \Theta) = f(\mathcal{L}) \leq f \circ e_\Sigma(a) = e_\Sigma(a)\} \quad (7.9 \text{ (ii) and since } f \text{ is order-preserving})$$
$$= \{\mathcal{J} \in \lambda_\Theta X: \mathcal{J} \leq e_\Theta(a)\} \quad (\text{since } f \text{ is surjective}).$$

Dually, if $R = X_I(a)$ for some $a \in X$, then

$$f[R^+] = \{\mathcal{J} \in \lambda_\Theta X: e_\Theta(a) \leq \mathcal{J}\}.$$

Thus the image under f of each member of Σ^+ is a closed subset of $\lambda_\Theta X$, and since f is injective, it follows that f is a closed map.

It follows from the preceding arguments that there is an iseomorphism $f: \lambda_\Sigma X \to \lambda_\Theta X$ such that $f \circ e_\Sigma = e_\Theta$. Thus Θ is equivalent with $\Theta \cup T = \Sigma$ (cf., 7.6).

In view of Proposition 7.10 we may now assume without loss of generality that, for our purposes, all admissible subbases contain T. The following theorem now follows immediately from Propositions 7.4 and 7.10.

THEOREM 7.11. *If Σ is any admissible subbase for X, then $\lambda_\Sigma X$ algebraically dominates $\lambda_T X$ and is algebraically dominated by $\lambda_\Gamma X$.*

We consider now some situations in which one lattice superextension of X dominates another lattice superextension of X not only algebraically, but also as a superextension.

PROPOSITION 7.12. *If Θ and Σ are admissible subbases for X such that $\Theta \subset \Sigma$ and Θ is weakly-normal, then*

(i) *for each $\mathcal{L} \in \lambda_\Sigma X$, $(\mathcal{L} \cap \Theta)$ is contained in a unique maximal linked system of Θ, and*

(ii) *the superextension $\lambda_\Sigma X$ dominates the superextension $\lambda_\Theta X$.*

PROOF. (i) Suppose that $\mathcal{L} \in \lambda_\Sigma X$ and $(\mathcal{L} \cap \Theta)$ is not contained in a unique maximal linked system of Θ. Then there exist $\mathcal{M}, \mathcal{N} \in \lambda_\Theta X$ such that $\mathcal{M} \neq \mathcal{N}$, $(\mathcal{L} \cap \Theta) \subset \mathcal{M}$ and $(\mathcal{L} \cap \Theta) \subset \mathcal{N}$. Since $\mathcal{M} \neq \mathcal{N}$, there exist $T \in \mathcal{M}$ and $S \in \mathcal{N}$ such that $S \cap T = \emptyset$ (2.16 (ii)). By the weak-normality of Θ, there is a finite screen $\{T_1, T_2, \cdots, T_n\} \subset \Theta$ of S and T (2.11 (iii)). Since $\beta_\Sigma X = \lambda_\Sigma X$, it follows that $\mathcal{L}$ is prime (2.18 (iii)) and hence there is some $j \in \{1, 2, \cdots, n\}$ such that $T_j \in \mathcal{L}$. Therefore, $T \cap T_j \neq \emptyset$ and $S \cap T_j \neq \emptyset$--a contradiction to the fact that $\{T_1, \cdots, T_n\}$ is a screen for S and T. Thus $(\mathcal{L} \cap \Theta)$ is contained in a unique maximal linked system of Θ.

(ii) By (i) above and Proposition 7.5 (i), there is a unique dominating function $f: \lambda_\Sigma X \to \lambda_\Theta X$ which may be defined by

$f(\mathcal{L}) = \{T \in \Theta: (\mathcal{L} \cap \Theta) \cup \{T\} \text{ is linked}\}$ for each $\mathcal{L} \in \lambda_\Sigma X$.

To see that f is continuous, let $T \in \Theta$ and $\mathcal{L} \in \lambda_\Sigma X$ such that $\mathcal{L} \notin f^{-1}[T^*]$. Thus $T \notin f(\mathcal{L})$ so that $\{T\} \cup (\mathcal{L} \cap \Theta)$ is not linked. Hence there is some $S \in (\mathcal{L} \cap \Theta)$ such that $T \cap S = \emptyset$. By the weak-normality of Θ, there is a finite screen $\Psi \subseteq \Theta$ of S and T. Let $C = \cup\{W^+: W \in \Psi \text{ and } W \cap S = \emptyset\}$. Clearly C is a closed subset of $\lambda_\Sigma X$. Since $S \in \mathcal{L}$ and $S \cap W = \emptyset$ implies that $S^+ \cap W^+ = \emptyset$ (2.16 (i)), it follows that $\mathcal{L} \notin W^+$ for each $W \in \Psi$ with $W \cap S = \emptyset$; i.e., $\mathcal{L} \notin C$. If $\mathcal{J}$ is any member of $f^{-1}[T^*]$, then since $\mathcal{J} \in \lambda_\Sigma X = \cup\{W^+: W \in \Psi\}$, there is some $\hat{W} \in \Psi$ such that $\mathcal{J} \in \hat{W}^+$; i.e., $\hat{W} \in \mathcal{J}$. Since $\hat{W} \in \Theta$, we see that $\hat{W} \in (\mathcal{J} \cap \Theta) \subseteq f(\mathcal{J})$ (7.2 (iii)), and since $T \in f(\mathcal{J})$, it follows that $T \cap \hat{W} \neq \emptyset$; therefore, $S \cap \hat{W} = \emptyset$. Thus $\mathcal{J} \in \hat{W}^+ \subseteq \cup\{W^+: W \in \Psi \text{ and } W \cap S = \emptyset\} = C$. It follows that $f^{-1}[T^*] \subseteq C$ and $\mathcal{L} \notin C$. Thus since $\mathcal{L}$ was an arbitrary point not in $f^{-1}[T^*]$, it follows that $f^{-1}[T^*]$ is a closed subset of $\lambda_\Sigma X$ and so f is continuous.

If Ψ is a non-void collection of subsets of X, then $\Psi^\cap$ denotes the collection of arbitrary intersections of members of Ψ.

PROPOSITION 7.13. *If Θ is an admissible subbase for X and $\Gamma = T_D^\cap \cup T_I^\cap$, then*

(i) *for each* $\mathcal{L} \in \lambda_\Theta X$, $(\mathcal{L} \cap T)$ *is a maximal linked system of* T, *and*

(ii) *the superextension* $\lambda_\Theta X$ *dominates the superextension* $\lambda_T X$.

PROOF. (i) Let $\mathcal{L} \in \lambda_\Theta X$. Clearly $(\mathcal{L} \cap T)$ is a linked system of T. If $U \in T_D$ such that $(\mathcal{L} \cap T) \cup \{U\}$ is linked, then either $U = X \in (\mathcal{L} \cap T)$ or there is some $a \in X$ such that $U = X_D(a)$. To see that $X_D(a) \in \mathcal{L}$, let $T \in \mathcal{L} \cap \Theta_I$. By the hypotheses, $T = \cap\{V \in T_I : T \subseteq V\}$. For each $V \in T_I$ such that $T \subseteq V$, it follows that $V \in (\mathcal{L} \cap T)$, $V \cap X_D(a) \neq \emptyset$, and $a \in V$; therefore $a \in T$ so that $T \cap X_D(a) \neq \emptyset$. Since T was an arbitrary member of $(\mathcal{L} \cap \Theta_I)$, it follows that $(\mathcal{L} \cap \Theta_I) \cup \{X_D(a)\}$ is linked and hence $\mathcal{L} \cup \{X_D(a)\}$ is linked. Consequently, $X_D(a) \in (\mathcal{L} \cap T)$. Dually, if $U \in T_I$ and $(\mathcal{L} \cap T) \cup \{U\}$ is linked, then $U \in (\mathcal{L} \cap T)$. Thus $(\mathcal{L} \cap T)$ is a maximal linked system of T.

(ii) That the superextension $\lambda_\Theta X$ dominates the superextension $\lambda_T X$ follows immediately from (i) above and Proposition 7.5 (ii).

We say that the lattice superextension $\lambda_\Sigma X$ is "larger" than the lattice superextension $\lambda_\Theta X$ (or, equivalently, $\lambda_\Theta X$ is "smaller" than $\lambda_\Sigma X$) if and only if the superextension $\lambda_\Sigma X$ dominates the superextension $\lambda_\Theta X$.

In the following theorem we sum up the preceding propositions obtaining two situations in which $\lambda_T X$ is actually the smallest lattice superextension of X. The proof is immediate from Propositions 7.12 and 7.13.

THEOREM 7.14. If T is weakly-normal or if $\Gamma = T_D^{\cap} \cup T_I^{\cap}$, then $\lambda_T X$ is the smallest lattice superextension of X; specifically, if Θ is any admissible subbase, then the superextension $\lambda_\Theta X$ dominates the superextension $\lambda_T X$.

In view of the relationship between T, Γ and any admissible subbase Σ, namely, $T \subseteq \Sigma \subseteq \Gamma$, it is natural to ask if $\lambda_T X$ is always the smallest or if $\lambda_\Gamma X$ is always the largest lattice superextension of X. In general the answer is negative to both questions. As noted previously there is a topological lattice X for which the unique dominating function $f: \lambda_\Gamma X \to \lambda_T X$ is not continuous; i.e., $\lambda_T X$ is not smaller than $\lambda_\Gamma X$ and $\lambda_\Gamma X$ is not larger than $\lambda_T X$ (8.7). Even under the restriction that $\Gamma = T_D^{\cap} \cup T_I^{\cap}$, $\lambda_\Gamma X$ is not necessarily the largest. In Example (8.6, Part II) there is a topological lattice which satisfies this restriction, but for which there is an admissible subbase Σ such that there is no continuous dominating function $f: \lambda_\Gamma X \to \lambda_\Sigma X$; i.e., $\lambda_\Gamma X$ is not the largest superextension of X.

The theorem below follows immediately from Proposition 7.12 (ii) and Theorem 7.14. It is stated here for completeness.

THEOREM 7.15. If all admissible subbases for X are weakly-normal, then $\lambda_\Gamma X$ is the largest lattice superextension of X and $\lambda_T X$ is the smallest lattice superextension of X.

DEFINITION 7.16. (Kaufman [12]) Y is called an orderable compactification of a totally ordered topological lattice X if Y is a topological compactification of X and the order on X can be extended to Y so as to yield the given topology on Y as the order (interval) topology.

COROLLARY 7.17. (Kaufman) If X is totally, then X has a largest and a smallest orderable compactification.

PROOF. Suppose that X is totally ordered. Thus each admissible subbase for X is weakly-normal (4.12). By Theorem 7.15, $\lambda_\Gamma X$ is the largest and $\lambda_T X$ is the smallest lattice superextension of X. It is shown in Appendix I that Kaufman's orderable compactifications of X are precisely the lattice superextensions of X; therefore, $\lambda_\Gamma X$ is the largest orderable compactification of X and $\lambda_T X$ is the smallest orderable compactification of X.

In view of Kaufman's method for obtaining orderable compactifications, we may characterize the lattice superextensions of a totally ordered space X as the topological closures of $h_E[X]$ in I^E where I is the closed interval [0,1], E is any subset of {f: f is a continuous increasing map of X into I such that $\wedge f[X] = 0$ and $\vee f[X] = 1$} which separates the points of X, and $h_E: X \to I^E$ is defined by $\pi_f(h_E(x)) = f(x)$ for each $f \in E$ and each $x \in X$.

8. EXAMPLES

EXAMPLE 8.1. Let $X = (0,\frac{1}{2}) \cup (\frac{1}{2},1)$ with the usual ordering. Then the completion of X by cuts, [0,1] with its usual ordering, is iseomorphic with $\lambda_T X$ (5.13), and the largest superextension $\lambda_\Gamma X$ is iseomorphic with $[0,\frac{1}{4}] \cup [\frac{3}{4},1]$ with its usual ordering. (The maximal linked system of Γ defined by $\{(0,\frac{1}{2})\} \cup \{(0,a]: \frac{1}{2} < a < 1\} \cup \{[a,1): 0 < a < \frac{1}{2}\}$ corresponds to the point $\frac{1}{4}$ in $\lambda_\Gamma X$, and the one defined by $\{(\frac{1}{2},1)\} \cup \{(0,a]: \frac{1}{2} < a < 1\} \cup \{[a,1): 0 < a < \frac{1}{2}\}$ corresponds to $\frac{3}{4}$ in $\lambda_\Gamma X$.)

EXAMPLE 8.2. Let X = (0,1) with its usual ordering. Then $T = \Gamma$ and consequently every lattice superextension of X is iseomorphic with [0,1] with its usual ordering.

EXAMPLE 8.3. Let n be a positive integer. For each $j \in \{1, \cdots, n\}$, let $I_j = (0,1)$ with its usual ordering. Thus each I_j is a topological lattice.

Let Y be the topological product of $\{I_j: j = 1, \cdots, n\}$, and for each $j = 1, \cdots, n$, let $\Sigma_j = T$. (Note that for each I_j, $T = \Gamma$.)

Thus $\Sigma = \{\pi_j^{-1}[T]: T \in \Sigma_j, j \in \{1, \cdots, n\}\}$ is a T_1-subbase for Y (3.1), and the superextension (de Groot compactification) of the product Y with respect to Σ, $\lambda_\Sigma Y$ ($\beta_\Sigma Y$), is homeomorphic with the product of the superextensions $\{\lambda_{\Sigma_j} I_j: j = 1, \cdots, n\}$ (3.3) (de Groot compactifications $\{\beta_{\Sigma_j} I_j: j = 1, \cdots, n\}$ (3.4)).

Since each $I_j = (0,1)$, it follows from Example 8.2 that $\lambda_\Sigma Y$ is homeomorphic with a closed n-cell in Euclidean n-space; therefore, a de Groot compactification of an open n-cell in Euclidean n-space is the n-cell with boundary attached. Note also that since the results in Chapter 3 (Products of Superextensions) were given for arbitrary products of T_1-spaces, we can generalize this example to the case that Y is the product of an arbitrary collection of copies of (0,1). In particular, in view of Anderson's result [3], a de Groot compactification of Hilbert space is the Hilbert cube.

EXAMPLE 8.4. Let I = [0,1] with its usual ordering, and let $X = \{x \in I: x \text{ is rational}\}$. Then under the usual ordering X is a topological lattice, and since I is the completion of X by cuts, $\lambda_T X$ is iseomorphic with I (5.13).

For each irrational $y \in I$, let y_ℓ and y_u be distinct points not in X which correspond to y. Let

$$Y = X \cup \{y_\ell: y \in I \text{ and } y \text{ is irrational}\} \cup \{y_u: y \in I \text{ and } y \text{ is irrational}\}.$$

We extend the ordering on X to one on Y in the following way:

(a) If $y \in I$ and y is irrational, then for every $x \in X$ and $x' \in X$ such that $x <_I y <_I x'$, we let $x <_Y y_\ell <_Y y_u <_Y x'$.

(b) If $y, z \in I$ are both irrational with $y <_I z$, then we let $y_u <_Y z_\ell$.

We prove the following statements:

(i) Y is a complete (totally ordered) lattice, and X is a sublattice of Y.

PROOF. It is easily seen from the definition of the ordering on Y that X is a sublattice of Y, and since I is totally ordered, it follows that Y is also totally ordered. To see that Y is complete, suppose that C is a non-void subset of Y. Let $C' = (C \cap X) \cup \{y \in I:$ y is irrational and either $y_\ell \in C$ or $y_u \in C\}$. Thus C' is a non-void subset of the complete lattice I so that there is some $c \in I$ such that $c = \vee_I C'$. If $c \in X$, it follows from the definitions that $c = \vee_Y C$ and $c \in Y$. If c is irrational and $c_u \in C$, then again from the definitions we see that $c_u = \vee_Y C \in Y$. Finally, if c is irrational and $c_u \notin C$, then $\vee_Y C = c_\ell \in Y$. A dual argument shows that $\wedge_Y C$ exists in Y.

(ii) $\lambda_\Gamma X$ is iseomorphic with Y.

PROOF. Define $i: X \to Y$ by $i(x) = x$ for each $x \in X$; i.e., i is the injection map of X into Y. Since X is a sublattice of Y, i is a

lattice inclusion. The equalities below follow readily from the definitions and some properties of the real numbers:

(a) If $y \in I$ is irrational, then
$i^{-1}[Y_D(y_\ell)] = i^{-1}[Y_D(y_u)] = \cap\{X_D(x): x \in X \text{ and } y \leq_I x\}$ and
$i^{-1}[Y_I(y_\ell)] = i^{-1}[Y_I(y_u)] = \cap\{X_I(x): x \in X \text{ and } x \leq_I y\}$.

(b) If $x \in X$, then $i^{-1}[Y_D(i(x))] = X_D(x)$ and $i^{-1}[Y_I(i(x))] = X_I(x)$.

It follows from (a) and (b) that i is continuous; i.e., i is a continuous lattice inclusion.

(c) $i[X]^1 = i[X]$, $i[X]^{1a} = \{y_\ell: y \in I \text{ and } y \text{ is irrational}\} \cup i[X]$, and $i[X]^{1b} = \{y_u: y \in I \text{ and } y \text{ is irrational}\} \cup i[X]$.

It follows from (c) that $Y = i[X]^{1a} \cup i[X]^{1b} \subseteq i[X]^2 \subseteq Y$, i.e., $Y = i[X]^2$. Thus Y is a topological lattice completion of X with respect to i (6.10).

To see that X is lattice dense in Y, suppose that $v, w \in Y$ with $v \in i[X]^{1a} - i[X]^1$, $w \in i[X]^{1b} - i[X]^1$, and $w <_Y v$. It follows from (c) that there exist $y, z \in I$ such that y and z are irrational with $w = y_u$ and $v = z_\ell$. Since $y_u <_Y z_\ell$, it must be the case that $y <_I z$ and so there is some rational number $x \in X$ such that $y <_I x <_I z$; i.e., $w = y_u <_Y i(x) <_Y z_\ell = v$. Thus X is lattice dense in Y (6.18). It now follows that Y is iseomorphic with $\lambda_\Sigma X$ where $\Sigma = \{i^{-1}[S]: S \in \Sigma^{(i)}\}$ (6.20). To see that Y is iseomorphic with $\lambda_\Gamma X$, we need only

show that $\Sigma = \Gamma$. Recall that

$$\Sigma^{(i)} = \{Y\} \cup \{Y_D(i(x)): x \in X\} \cup \{Y_I(i(x)): x \in X\} \cup$$
$$\{Y_D(y_\ell): y \in I \text{ and } y \text{ is irrational}\} \cup$$
$$\{Y_I(y_u): y \in I \text{ and } y \text{ is irrational}\} \quad (6.15).$$

If T is a closed ideal of X, then either $T = X_D(x) = i^{-1}[Y_D(i(x))]$ for some $x \in X$ or $T = \{x \in X: x \leq_I y\} = i^{-1}[Y_D(y_\ell)]$ for some irrational $y \in I$. In either case it is clear that $T \in \Sigma_D$. A dual argument shows that $T \in \Sigma_I$ whenever T is a closed dual ideal of X. Thus $\Sigma = \Gamma$ and Y is iseomorphic with $\lambda_\Gamma X$.

EXAMPLE 8.5. Let A, B, C be subsets of the plane defined as follows:

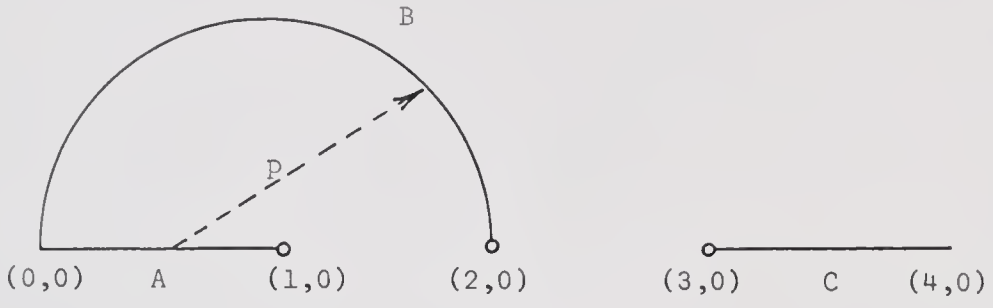
$$A = [0,1) \times \{0\}$$
$$B = \{(x,y): (x-1)^2 + y^2 = 1,\ 0 \leq x < 2,\ 0 \leq y \leq 1\}$$
$$C = (3,4] \times \{0\}.$$

Since A, B, and C are copies of half-open intervals we may assign to each of them a natural (total) ordering from left to right. Also since A and B are both copies of [0,1), there is a natural lattice isomorphism $p: A \to B$. Let $X = A \cup B \cup C$. X becomes a lattice by defining a supremum and an infimum for each pair of elements of X. Let $z, w \in X$. We let

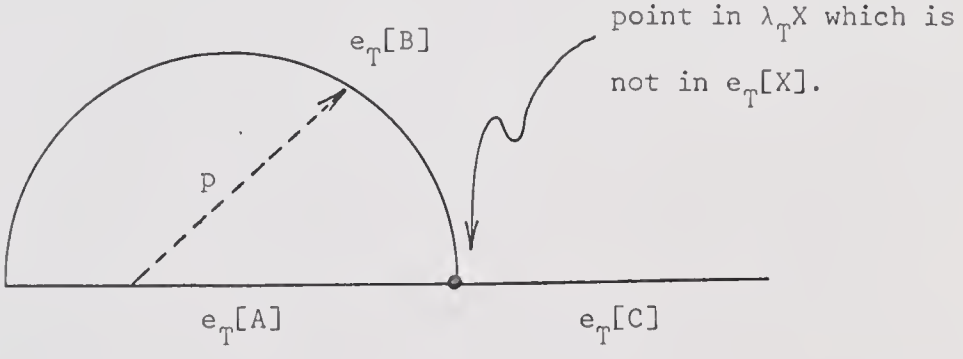
$$z \vee w = w \vee z = \begin{cases} p(w) \vee z, & \text{if } w \in A \text{ and } z \in B \\ w \vee z & \text{if } w, z \in A,\ w, z \in B, \text{ or } w, z \in C \\ z & \text{if } w \in A \cup B \text{ and } z \in C \end{cases}$$

$$z \wedge w = w \wedge z = \begin{cases} w \wedge p^{-1}(z), & \text{if } w \in A \text{ and } z \in B \\ w \wedge z, & \text{if } w, z \in A, w, z \in B, \text{ or } w, z \in C \\ w, & \text{if } w \in A \cup B \text{ and } z \in C. \end{cases}$$

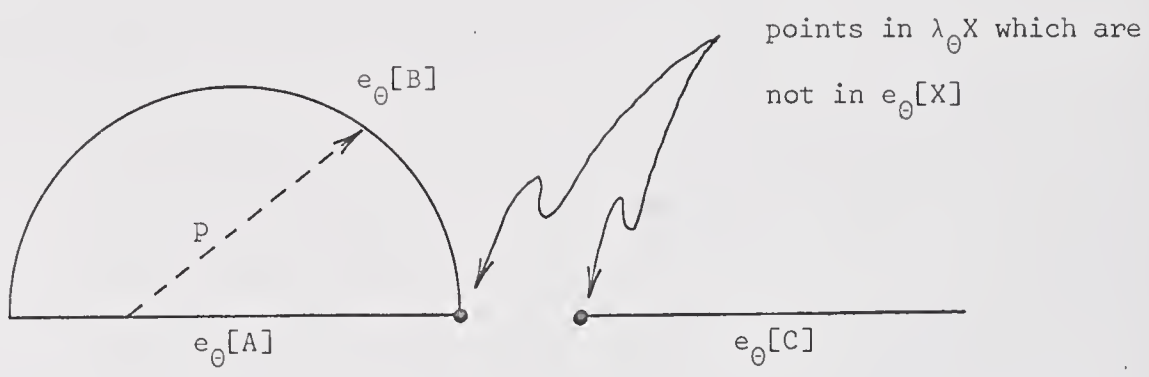
(i) The lattice described above can be visualized as follows:



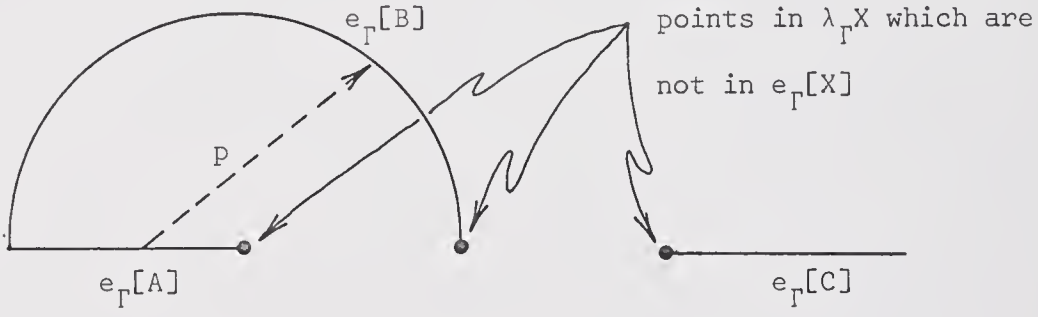
(ii) $\lambda_T X$ can be visualized as follows:



(iii) If $\Theta = T_D^{\cap} \cup T_I^{\cap}$, then $\Theta = T \cup \{A \cup B, C\}$ and $\lambda_\Theta X$ can be visualized as follows:



(iv) $\Gamma = T \cup \{A, A \cup B, C\}$ and $\lambda_\Gamma X$ can be visualized as follows:



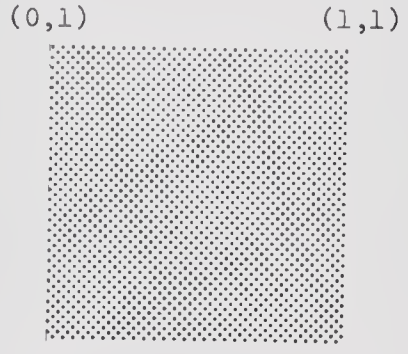
From (ii), (iii), and (iv) we see that three quite natural subbases yield different lattice superextensions.

EXAMPLE 8.6. Suppose that L and M are lattices. We define the lattice direct product, X, of L and M to be the set L × M with the

ordering defined by $(a,b) \leq_X (c,d)$ if and only if $a \leq_L c$ and $b \leq_M d$. It is easily seen that the lattice direct product is indeed a lattice.

Part I

Let X be the lattice direct product $(0,1) \times (0,1)$ where $(0,1)$ is the open interval with the usual ordering. Let Y_1 be the lattice direct product $[0,1] \times [0,1]$, where $[0,1]$ is the closed interval with the usual ordering. X may be visualized as follows:



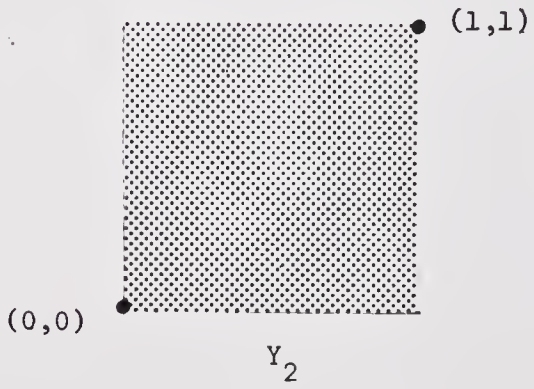
X

(i) X is a sublattice of Y_1 and $Y_1 = X^2 = X^{2a} \cap X^{2b}$, but the inclusion map $i: X \to Y_1$ is not continuous. Thus Y_1 is not a topological lattice completion of X with respect to i.

PROOF. It is easily seen from the definition of lattice direct product that X is a sublattice of Y_1, and it follows readily that $Y_1 \subseteq X^{1a} \cup X^{1b} \subseteq X^2 \subseteq Y_1$ so that $Y_1 = X^2$ $(= X^{2a} \cap X^{2b})$.

To see that i is not continuous, let $W = i^{-1}[Y_{1D}((1,\frac{1}{2}))]$. We will show that W is not a closed subset of X. Suppose that $r \in X$ such that $r \notin W$, and let V be any basic open (in the topology on X generated by $\{X - U: U$ is in the usual subbase for $X\}$) set containing r. It follows from the definitions that $V = A \cap B$ where $A = X - \cup\{X_D((a_i,b_i)): i = 1, 2, \cdots, n\}$ and $B = X - \cup\{X_I((c_j,d_j)): j = 1, 2, \cdots, m\}$ for some finite set $\{(a_1,b_1), \cdots, (a_n,b_n), (c_1,d_1), \cdots, (c_m,d_m)\} \subseteq X$. Let $a = \vee \{a_i: i = 1, 2, \cdots, n\}$ and $d = \wedge \{d_j: j = 1, 2, \cdots, m\}$. By the finiteness of n and m and the properties of (0,1), it is possible to find $(p,q) \in X$ where $a < p < 1$ and $0 < q < d \wedge \frac{1}{2}$. It follows that $(p,q) \in W \cap A \cap B = W \cap V$. Since V was an arbitrary basic open set about r, it follows that $r \in \overline{W}^X - W$; consequently, W is not closed in X.

Let $Y_2 = X \cup\{(0,0), (1,1)\}$. Extend the order of X to Y_2 by letting $(0,0) = \wedge_Y X$ and $(1,1) = \vee_Y X$. Y_2 may be visualized as follows:



(ii) The topological lattice Y_2 is iseomorphic with $\lambda_T X$.

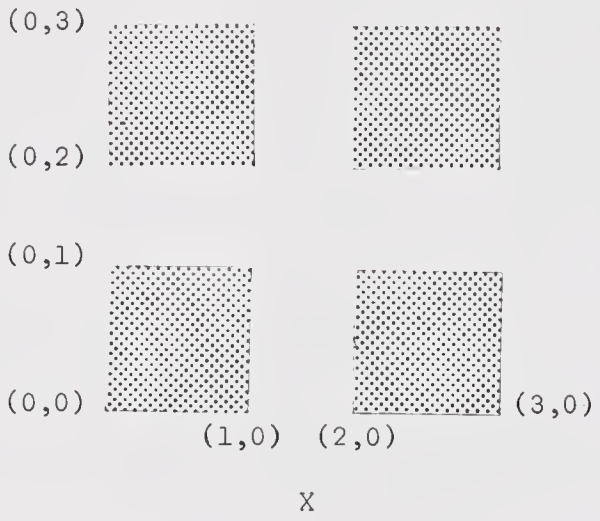
PROOF. It is easily seen that X is a sublattice of Y_2 and $Y_2 = X^{1a} \cap X^{1b} = X^1$. Thus Y_2 is the completion of X by cuts and so Y_2 is iseomorphic with $\lambda_T X$ (5.13 and 6.13).

(iii) In this example $\Gamma = T$; consequently, Y_2 is the unique lattice superextension of X.

PROOF. To see that $\Gamma = T$, let C be a closed ideal of X; i.e., $C \in \Gamma_D$. If C = X, then $C \in T_D$. If $C \neq X$, then since C is a closed subset of X, it follows as in the proof of (i) above that C cannot be of the form $\{(x,y) \in X: x \leq a \text{ for some } a \in (0,1)\}$ or of the form $\{(x,y) \in X: y \leq b \text{ for some } b \in (0,1)\}$. Thus there must be some $(a,b) \in X$ such that $C \subseteq X_D((a,b))$. Now by the completeness of the real numbers, it is possible to find real numbers c and d such that $c = \vee \{x: (x,y) \in C\}$ and $d = \vee \{y: (x,y) \in C\}$. Since $C \neq \emptyset$ and $C \subseteq X_D((a,b))$, it follows that $0 < c \leq a < 1$ and $0 < d \leq b < 1$. Hence $(c,d) \in X$. It is clear from the choice of c and d that $C \subseteq X_D((c,d))$. Since C is closed in the interval topology on X, and consequently in the order topology on X [9 ; Theorem 12], it follows that $(c,d) \in C$. Hence $C = X_D((c,d)) \in T_D$. A dual argument shows that $\Gamma_I \subseteq T_I$. Hence $\Gamma = T$, and since if Σ is any admissible subbase for X, we may assume without loss of generality that $T \subseteq \Sigma$ (7.10), it follows that $\Sigma = T$ and so $\lambda_\Sigma X = \lambda_T X$ is iseomorphic with Y_2.

Part II

In this part, let X be the lattice direct product $[(0,1)\cup(2,3)] \times [(0,1)\cup(2,3)]$ where $(0,1)\cup(2,3)$ has the usual order. X may be visualized as follows:



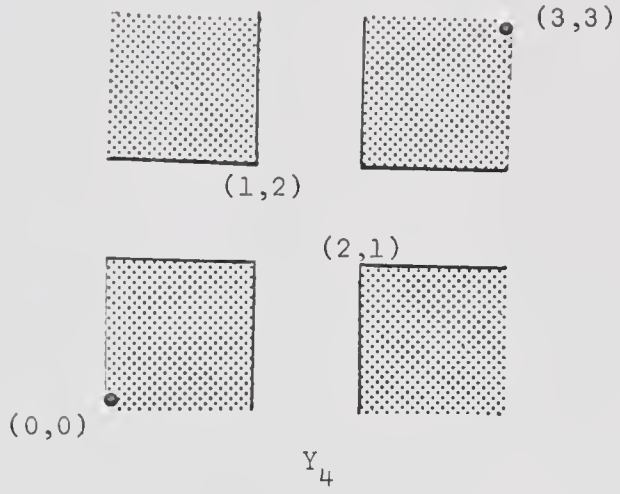
X

In this example it is easily seen that $\Gamma = T_D^{\cap} \cup T_I^{\cap}$.

(i) The lattice superextension $\lambda_T X$ naturally fills in the deleted cross so that $\lambda_T X$ is iseomorphic with the topological lattice Y_2 (Part I).

Let Y_3 be the lattice direct product $[\,(0,1] \cup [2,3)\,] \times [\,(0,1] \cup [2,3)\,]$, and let $Y_4 = Y_3 \cup \{(0,0), (3,3)\}$. Extend the ordering of Y_3 to Y_4 by letting $(0,0) = \wedge\, Y_3$ and $(3,3) = \vee\, Y_3$.

Thus the map $i: X \to Y_4$ defined by $(x,y) \mapsto (x,y)$ is a continuous lattice inclusion and so X is a sublattice of Y_4. Y_4 can be visualized as follows:



(ii) Y_4 is a topological lattice completion of X with respect to i.

PROOF. Since i is a continuous lattice inclusion, it follows from the equalities below that Y_4 is a topological lattice completion of X with respect to i (6.10):

$$i[X]^1 = X \cup \{(0,0), (3,3)\},$$

$$i[X]^{1a} - i[X]^1 = \{(x,y): x = 1 \text{ or } y = 1\} - \{(1,2), (2,1)\},$$

$$i[X]^{1b} - i[X]^1 = \{(x,y): x = 2 \text{ or } y = 2\} - \{(1,2), (2,1)\}, \text{ and}$$

$$i[X]^2 - (i[X]^{1a} \cup i[X]^{1b}) = \{(1,2), (2,1)\}.$$

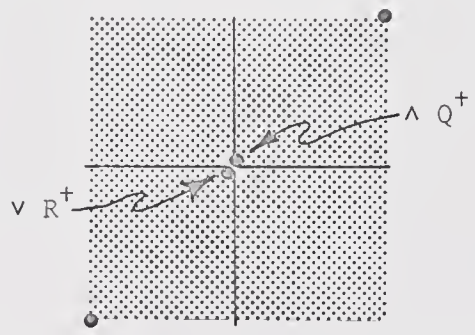
Thus $Y_4 = i[X]^2$.

(iii) $\lambda_\Gamma X$ is iseomorphic with the topological lattice Y_4.

PROOF. Since each proper non-principal closed ideal of X is of the form $i^{-1}[Y_{4D}(\ (a,b)\ )]$ where a = 1 or b = 1 and each proper non-principal closed dual ideal of X is of the form $i^{-1}[Y_{4I}(\ (a,b)\ )]$ where a = 2 or b = 2, it follows that $\Gamma = \{i^{-1}[S]: S \in \Sigma^{(i)}\}$. It is easily seen (using the equalities in (ii)) that X is lattice dense in Y_4 (6.18). Thus Y_4 is iseomorphic with $\lambda_\Gamma X$ (6.20).

Let $\Sigma = T \cup \{R,Q\}$ where $R = i^{-1}[Y_{4D}(\ (1,1)\ )]$ and $Q = i^{-1}[Y_{4I}(\ (2,2)\ )]$. Since i is a continuous lattice inclusion, it follows that Σ is an admissible subbase for X (4.3).

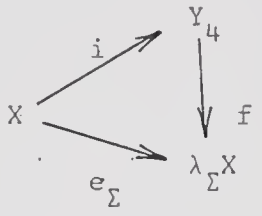
The lattice superextension $\lambda_\Sigma X$ can be visualized as follows:



$\lambda_\Sigma X$

(iv) There is no continuous algebraic dominating function $f: \lambda_\Gamma X \to \lambda_\Sigma X$.

PROOF. In order to simplify the notation in this proof, we recall that $\lambda_\Gamma X$ is iseomorphic with Y_4 and show that there is no continuous dominating function from Y_4 onto $\lambda_\Sigma X$. To do this, let f be any algebraic dominating function from Y_4 onto $\lambda_\Sigma X$. Thus we have the following commutative diagram:



Let p be the point (1,2) in Y_4. We now establish a few needed lemmas.

(1) Either $f(p) = \vee R^+$ or $f(p) = \wedge Q^+$: For each $(x,y) \in R$, $i((x,y)) \leq (1,1) \leq (1,2) = p$ and since f is order preserving, it follows that $f(i((x,y))) = e_\Sigma((x,y)) \leq f(p)$. Thus $\vee R^+ = \vee e_\Sigma[R] \leq f(p)$. A dual argument shows that $f(p) \leq \wedge Q^+$. Now, from the definition of Σ it follows that there is no $\mathcal{L} \in \lambda_\Sigma X$ such that $\vee R^+ < \mathcal{L} < \wedge Q^+$; consequently $\vee R^+ = f(p)$ or $\wedge Q^+ = f(p)$.

(2) $p \in \overline{\{(1,y): 2 < y < 3\}}^{Y_4} \cap \overline{\{(x,2): 0 < x < 1\}}^{Y_4}$: It follows from the definition of the order on Y_4 that $\vee_{Y_4} \{(x,2): 0 < x < 1\} = (1,2) = p = \wedge_{Y_4} \{(1,y): 2 < y < 3\}$. Hence by [9; Theorem 12], $p \in \overline{\{(x,2): 0 < x < 1\}}^{Y_4}$ and $p \in \overline{\{(1,y): 2 < y < 3\}}^{Y_4}$.

(3) $\{(1,y): 2 < y < 3\} \subseteq f^{-1}[Q^+]$ and $\{(x,2): 0 < x < 1\} \subseteq f^{-1}[R^+]$: Let $2 < y < 3$. It follows from the definition of Σ and the properties on f that $f(1,y) = \wedge_{\lambda_\Sigma X} \{e_\Sigma((a,b)): (1,y) \leq_{Y_4} i((a,b))\}$. Also, if $(a,b) \in X$

such that $(1,y) <_{Y_4} i((a,b))$, then $1 < a$ and $2 < y \leq b$. Thus $(2,2) \leq i((a,b))$ and so $(a,b) \in Q$. (Recall that there is no element $(a,b) \in X$ with $1 < a < 2$.) Hence for each $(a,b) \in X$ with $(1,y) \leq_{Y_4} i((a,b))$, it follows that $\wedge Q^+ \leq e_\Sigma((a,b))$; therefore $\wedge Q^+ \leq \wedge_{\lambda_\Sigma X} \{e_\Sigma((a,b)): (1,y) \leq_{Y_4} i((a,b))\} = f(1,y)$. Thus $Q \in f(1,y)$ (4.10 (vi)) and so $(1,y) \in f^{-1}[Q^+]$. It follows that $\{(1,y): 2 < y < 3\} \subseteq f^{-1}[Q^+]$. A dual argument shows that $\{(x,2): 0 < x < 1\} \subseteq f^{-1}[R^+]$.

Now by (1), either $f(p) = \vee R^+$ or $f(p) = \wedge Q^+$. If $f(p) = \vee R^+$, then $f(p) \notin Q^+$ and so $p \notin f^{-1}[Q^+]$, but it follows from (2) and (3) that $p \in \overline{f^{-1}[Q^+]}^{Y_4}$. Hence $f^{-1}[Q^+]$ is not a closed subset of X and so f is not continuous. In the case that $f(p) = \wedge Q^+$, it follows dually that $f^{-1}[R^+]$ is not a closed subset of X and so f is not continuous.

EXAMPLE 8.7. In this example we exhibit a topological lattice X for which $\lambda_\Gamma X$ algebraically dominates $\lambda_T X$ via a unique dominating function, but the dominating function is not continuous.

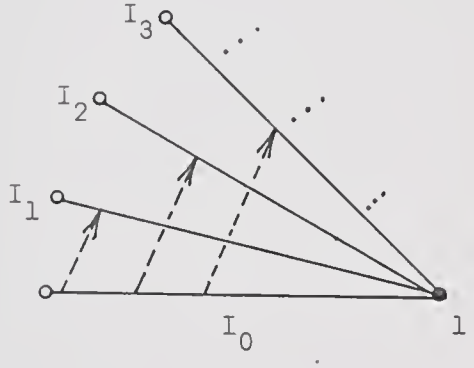
Let $N = \{0,1,2,3,\cdots\}$, and for each $n \in N$ let I_n be a copy of $(0,1]$ with its usual ordering. Denote the members of I_n by x_n where $x \in (0,1]$. Let X be the disjoint union of the I_n's for $n \in N$ modulo a relation R defined as follows: $x_n \; R \; x_{n'}'$ if and only if $n = n'$ and $x = x'$, or $x = x' = 1$. (We may now say that $1 \in X$ rather than $1_n \in X$ for each $n \in N$.) We define an ordering on X by assigning a supremum and an infimum to each pair of elements of X.

Let $x_n, x'_m \in X$.

$$x_n \vee x'_m = x'_m \vee x_n = \begin{cases} (x \vee x')_n, \text{ in case } n = m \\ (x \vee x')_m, \text{ in case } n \neq m \text{ and } n = 0 \\ 1, \text{ in all other cases} \end{cases}$$

$$x_n \wedge x'_m = x'_m \wedge x_n = \begin{cases} (x \wedge x')_n, \text{ in case } n = m \text{ or } x' = 1 \\ (x \wedge x')_0, \text{ in case } n \neq m. \end{cases}$$

X can be visualized as in the following diagram where the arrows denote lattice isomorphisms from I_0 onto each I_n:



(i) For each $n \in N - \{0\}$, I_n is a closed dual ideal of X.

PROOF. Suppose $y_m \in X$ and $x_n \leq y_m$ for some $x_n \in I_n$. Then $y_m = x_n \vee y_m$ and since $n \neq 0$ it follows from the definition of suprema in X that either $n = m$ or $y_m = 1$. In either case $y_m \in I_n$ and I_n is therefore increasing. It also follows from the definition of infima in X that I_n contains $x_n \wedge y_m$ when $x_n, y_m \in I_n$. To see that I_n is a

closed subset of X, suppose that $y_m \notin I_n$. Thus $n \neq m$ and $y \neq 1$. If $m = 0$, then we may pick $x \in (0,1)$ such that $0 < x < y$. It is easily seen that $I_n \subseteq X_D(x_n) \cup X_I(x_n)$ and $y_m \notin X_D(x_n) \cup X_I(x_n)$. If $m \neq 0$, then y_m is not comparable with any member of $I_n - \{1\}$ so let $x \in (0,1)$. Again it is easily seen that $I_n \subseteq X_D(x_n) \cup X_I(x_n)$ and $y_m \notin X_D(x_n) \cup X_I(x_n)$.

(ii) If D is a closed ideal of X, then D is a principal ideal.

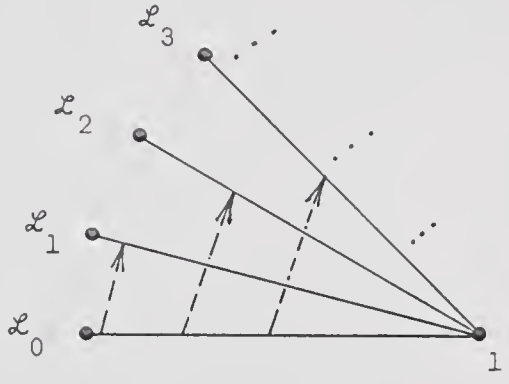
PROOF. If $D = X$, then $D = X_D(1)$. If $D \neq X$, then it follows from the definition of suprema in X and the fact that D contains $d_1 \vee d_2$ whenever $d_1, d_2 \in D$ that $D \subseteq I_0 \cup I_n$ for some $n \in N$. Since $D \cap I_n \neq \emptyset$, $D \cap I_n$ is a non-void closed ideal of I_n, and since $1 \notin D \cap I_n$, it follows from the completeness of the reals that there is some $x \in (0,1)$ such that $x_n = \vee_{I_n} (D \cap I_n)$. Since $D \cap I_n$ is a closed subset of X, it follows that $x_n \in D \cap I_n$ [9; Theorem 12]. Thus $X_D(x_n) \subseteq D$. If $d \in D$, then either $d \in I_n$, in which case $d \leq x_n$, or $d \in I_0$, in which case there is some $x' \in (0,1]$ such that $d = x_0'$. Thus $d = x_0' \leq x_0' \vee x_n = (x' \vee x)_n \in D \cap I_n$, and so $d \leq (x' \vee x)_n \leq x_n$; therefore, $D \subseteq X_D(x_n)$ and so $D = X_D(x_n)$.

(iii) If D is a proper closed dual ideal of X, then D is either a principal dual ideal or $D = I_n$ for some $n \in N - \{0\}$.

PROOF. Suppose that D is a closed dual ideal of X which is not principal and not equal to I_n for any $n \in N - \{0\}$. Thus $D \nsubseteq I_n$ for any $n \in N - \{0\}$, and so D meets at least two of the I_n's for $n \in N - \{0\}$; therefore, $D \cap I_0 \neq \emptyset$. Since D is not principal, it follows that $I_0 \subseteq D$. Since D is increasing, $X \subseteq D$; i.e., $D = X$ so that D is not proper.

In view of (i), (ii), and (iii), $\Gamma = T \cup \{I_n : n \in N - \{0\}\}$.

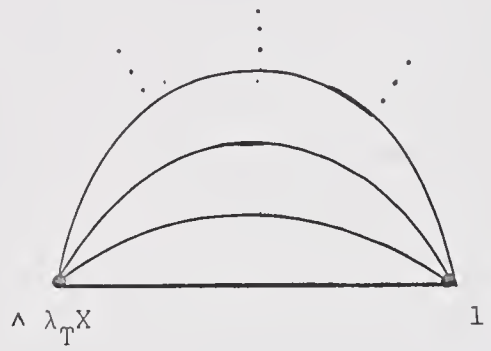
Let $\mathcal{L}_n = \wedge_{\lambda_\Gamma X} I_n^+$ for each $n \in N$. Since for each pair $m, n \in N - \{0\}$, $I_n \not\subseteq I_m$ and $I_m \not\subseteq I_n$, it follows that no two members of $\{\mathcal{L}_n : n \in N - \{0\}\}$ are comparable. Also, since $\mathcal{L}_0 = \wedge\, e_\Gamma[X] = \wedge\, \lambda_\Gamma X$, it follows that $\mathcal{L}_0 \leq \mathcal{L}_n$ for each n. $\lambda_\Gamma X$ may be visualized as follows:



(iv) For each $\mathcal{L} \in \lambda_\Gamma X$, either $\mathcal{L} \in e_\Gamma[X]$ or $\mathcal{L} = \wedge\, I_n^+$ for some $n \in N$.

Turning now to $\lambda_T X$ we note that for each $n \in N$, the only member of T_I which contains I_n is X. Thus it follows from Theorem 4.8

that for each n, $\wedge e_T[I_n] = \{X\} \cup T_D$ and hence $\wedge e_T[I_n] = \wedge \lambda_T X$. The lattice $\lambda_T X$ may be visualized as follows:



(v) If $\mathcal{L} \in \lambda_T X$, then $(\mathcal{L} \cap T)$ is contained in a unique maximal linked system of T.

PROOF. Let $\mathcal{L} \in \lambda_T X$. If $\mathcal{L} \in e_T[X]$, then $(\mathcal{L} \cap T)$ is a maximal linked system of T. If $\mathcal{L} \notin e_T[X]$, then by (iv), $\mathcal{L} = \wedge I_n^+ = \mathcal{L}_n$ for some $n \in N$. Thus, as was noted above, the only member of T_I containing I_n is X, and it follows that $(\mathcal{L}_n \cap T) = \{X\} \cup \{X_D(x_n): x_n \in I_n\} \subseteq \{X\} \cup T_D = \wedge \lambda_T X$. If $\mathcal{M} \in \lambda_T X$ such that $\wedge \lambda_T X < \mathcal{M}$, then there is a principal dual ideal $T \in (\mathcal{M} \cap T_I) - (\wedge \lambda_T X)$ (4.7). Clearly $I_n \not\subseteq T$, so there is some $r_n \in I_n - T$. Hence $X_D(r_n) \in (\mathcal{L}_n \cap T)$; therefore $(\mathcal{L}_n \cap T)$ cannot be contained in $\mathcal{M}$. It follows that for each $\mathcal{L} \in \lambda_T X$, $(\mathcal{L} \cap T)$ is contained in a unique maximal linked system of T.

In view of (v), $\lambda_\Gamma X$ algebraically dominates $\lambda_T X$ via a unique dominating function $f: \lambda_\Gamma X \to \lambda_T X$ which may be defined by

$$f(\mathcal{L}) = \{U \in T: (\mathcal{L} \cap T) \cup \{U\} \text{ is linked}\}$$

for each $\mathcal{L} \in \lambda_\Gamma X$ (7.5 (i)).

(vi) The dominating function $f: \lambda_\Gamma X \to \lambda_T X$ is not continuous.

PROOF. Let $t \in (0,1)$. Then $t_0 \in I_0 \subseteq X$, and $X_D(t_0) \in T_D$. We will show that $f^{-1}[(X_D(t_0))^*]$ is not a closed subset of $\lambda_\Gamma X$: To see this, let $r \in (t,1)$. It is clear that $e_\Gamma(r_0) \notin f^{-1}[(X_D(t_0))^*]$ and we will show that there is no finite subset of Γ^+ whose union covers $f^{-1}[(X_D(t_0))^*]$ and misses $e_\Gamma(r_0)$. This follows from the following statements:

(1) For each $n \in N$, $\mathcal{L}_n \in f^{-1}[(X_D(t_0))^*]$: If $n \in N$, then as was shown in the proof of (v) above, $f(\mathcal{L}_n) = \wedge \lambda_T X \leq \vee (X_D(t_0))^*$. Thus $X_D(t_0) \in f(\mathcal{L}_n)$ (4.10 (vi)).

(2) If $n, m \in N - \{0\}$ and $n \neq m$, then $\mathcal{L}_n \vee \mathcal{L}_m = e_\Gamma(1)$: Clearly $\mathcal{L}_n \vee \mathcal{L}_m \leq e_\Gamma(1) = \vee \lambda_\Gamma X$. If $S \in \mathcal{L}_n \vee \mathcal{L}_m \cap \Gamma_D$, then by (ii) above $S = X_D(x_k)$ for some $x_k \in X$, and it must be the case that $S \cap I_n \neq \emptyset$ and $S \cap I_m \neq \emptyset$. Hence $x_k \in I_n \cap I_m$; i.e., $x_k = 1$ and $S = X$. Thus $S \in e_\Gamma(1)$ and so $e_\Gamma(1) \leq \mathcal{L}_m \vee \mathcal{L}_n$. Hence $e_\Gamma(1) = \mathcal{L}_n \vee \mathcal{L}_m$.

(3) If $X_D(x_k) \in \mathcal{L}_n$ and $n \neq 0$, then $k = n$ or $x = 1$: If $X_D(x_k) \in \mathcal{L}_n$, then $X_D(x_k) \cap I_n \neq \emptyset$ and $x_k \in I_n$. Thus $k = n$ or $x = 1$.

(4) If $S \in \Gamma_D$ and $f^{-1}[(X_D(t_0))^*] \subseteq S^+$, then $S = X$: Suppose that $S \in \Gamma_D$ and $f^{-1}[(X_D(t_0))^*] \subseteq S^+$. By (1), $\mathcal{L}_n \in S^+$ for each $n \in N$. Thus by (2) and since S^+ is an ideal of $\lambda_\Gamma X$ (4.10 (i)), $e_\Gamma(1) \in S^+$; therefore, $S = X_D(1) = X$.

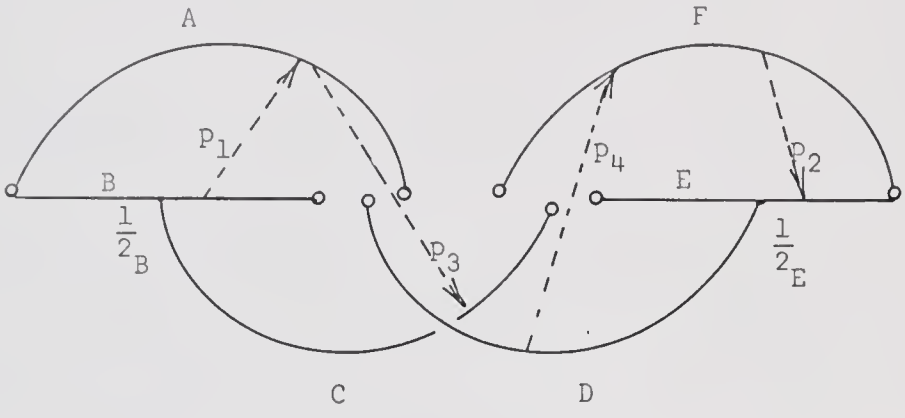
(5) If $S \in \Gamma_D - \{X\}$, then S^+ contains at most two members of $\{\mathcal{L}_n : n \in N\}$: If $S \in \Gamma_D - \{X\}$, then by (ii) above $S = X_D(x_k)$ for some $x_k \in X$. Since $S \neq X$, $x_k \neq 1$, and so it follows from (3) that $S^+ \cap \{\mathcal{L}_n : n \in N\} = \{\mathcal{L}_0, \mathcal{L}_k\}$.

(6) If $S \in \Gamma_I - \{X\}$, then S^+ contains at most one member of $\{\mathcal{L}_n : n \in N\}$: Suppose that $S \in \Gamma_I - \{X\}$. By (iii) either $S = X_I(x_k)$ for some $x_k \in X$ or $S = I_k$ for some $k \in N - \{0\}$. In the first case $S^+ \cap \{\mathcal{L}_n : n \in N\} = \emptyset$ and in the second case $S^+ \cap \{\mathcal{L}_n : n \in N\} = \{\mathcal{L}_k\}$.

We have shown in (5) and (6) that each member of $(\Gamma - \{X\})^+$ contains at most two members of $\{\mathcal{L}_n : n \in N\}$. Since $\{\mathcal{L}_n : n \in N\}$ is an infinite subset of $f^{-1}[(X_D(t_0))^*]$, it follows that no finite subcollection of Γ^+ can cover $f^{-1}[(X_D(t_0))^*]$ and miss $e_\Gamma(r_0)$. Thus $f^{-1}[(X_D(t_0))^*]$ is not closed, and so f is not continuous.

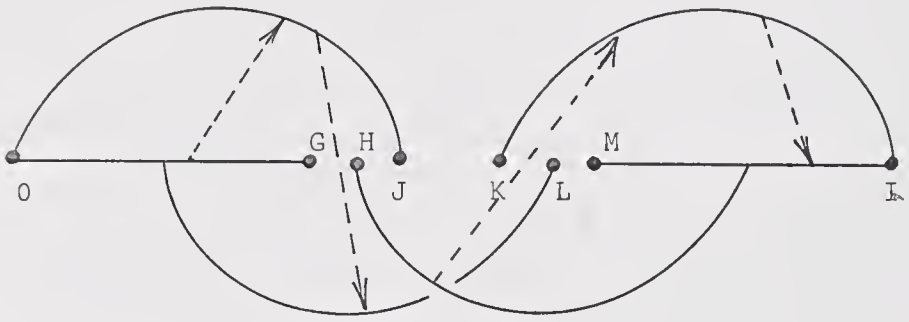
EXAMPLE 8.8. Here we exhibit a topological lattice which has a topological lattice completion in which the given lattice is not lattice dense (6.18). Let each of A, B, C, D, E, and F be the

interval (0,1) with its usual ordering. Let X be made up of these six pieces in the manner shown in the following diagram where the p_i's, i = 1,2,3,4, are lattice isomorphisms between the various parts:



All elements of B are less than all elements of D $\cup$ E $\cup$ F. All elements of A are less than all elements of F $\cup$ E. All elements of C are less than all elements of E. An element $b \in B$ is less than $p_1(b) \in A$, and an element $a \in A$ is less than $p_3(a) \in C$. Dually, an element $d \in D$ is less than $p_4(d) \in F$, and an element $f \in F$ is less than $p_2(f) \in E$. Finally, note that $\wedge_X C = \frac{1}{2}_B$ and $\vee_X D = \frac{1}{2}_E$.

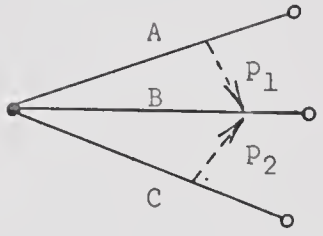
Let Y contain X as in the following diagram:



We extend the order on X to Y by letting $0 = \wedge_Y X = \wedge Y$, $I = \vee_Y X = \vee Y$, and $G \leq H \leq J \leq K \leq L \leq M$. It is easily seen that $Y = X^{1a} \cup X^{1b} = X^2$ (6.1) and the identity map $i: X \to Y$ is a continuous lattice inclusion. Thus Y is a topological lattice completion of X with respect to i (6.10). It is clear from the diagram that $H \in {}_i[X]^{1b} - {}_i[X]^1$, $J \in {}_i[X]^{1a} - {}_i[X]^1$, and $H < J$. However, $\{x \in X: H < i(x) < J\} = \emptyset$; i.e., X is not lattice dense in Y (6.18).

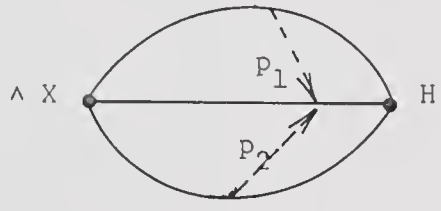
EXAMPLE 8.9. In this example we exhibit a topological lattice X for which $\lambda_\Gamma X$ algebraically dominates $\lambda_T X$ (and even dominates it as a superextension), but for which the dominating function of $\lambda_\Gamma X$ over $\lambda_T$is not a lattice homomorphism (2.2 (i)).

Let A, B, and C be copies of [0,1) with the usual ordering, and let X be made up of A, B, and C as in the following diagram where p_1 and p_2 are lattice isomorphisms between the indicated parts:

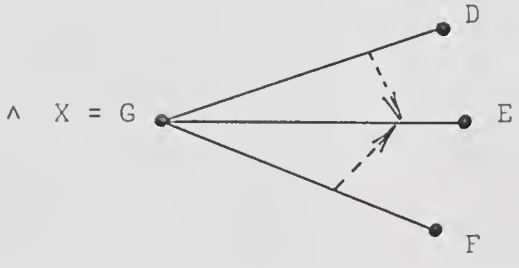


An element $a \in A$ is less than $p_1(a) \in B$, and an element $c \in C$ is less than $p_2(c) \in B$.

(i) $\lambda_T X$ can be visualized as follows:



(ii) $\Gamma = T \cup \{A,C\}$ and $\lambda_\Gamma X$ can be visualized as follows:



It is easily seen that T is a weakly-normal admissible subbase for X. Thus the superextension $\lambda_\Gamma X$ dominates the superextension $\lambda_T X$ via a unique dominating function $f: \lambda_\Gamma X \to \lambda_T X$ (7.5 and 7.12).

(iii) The dominating function $f: \lambda_\Gamma X \to \lambda_T X$ is not a lattice homomorphism.

PROOF. The points D and F in $\lambda_\Gamma X$ are not in $e_\Gamma[X]$, and by Theorem 4.8, $D \wedge F = G = \wedge X$. Since clearly $f(D) = f(F) = H$ in $\lambda_T X$, it follows that $f(D) \wedge f(F) = H$; therefore,

$$f(D \wedge_{\lambda_\Gamma X} F) \neq f(D) \wedge_{\lambda_T X} f(F).$$

Hence f is not a lattice homomorphism (cf., 2.2 (i)).

APPENDIX I

Kaufman's Orderable Compactifications

The purpose of this Appendix is to show that for totally ordered topological lattices, Kaufman's [12] orderable compactifications (7.16) are precisely the lattice superextensions.

Throughout this Appendix X will denote a totally ordered topological lattice. Frink [9 ; Theorem 3] has shown that the order and interval topologies coincide for totally ordered spaces. Kaufman [12; Theorem 2] has shown that the orderable compactifications of X are also totally ordered spaces; hence the order and the interval topologies in an orderable compactification of X coincide.

PROPOSITION I(1). *Every lattice superextension of* X *is an orderable compactification of* X.

PROOF. Let Σ be any admissible subbase for X. Then $e_\Sigma[X]$ is iseomorphic with X (2.16 (v) and 4.9) so that we may identify X with $e_\Sigma[X]$. Since $\lambda_\Sigma X = \beta_\Sigma X$, X is topologically dense in $\lambda_\Sigma X$, and since e_Σ is a lattice inclusion (4.9), it follows that the ordering on $\lambda_\Sigma X$ is an extension of that on X. Finally, we recall that the topology on $\lambda_\Sigma X$ generated by Σ^+ is the interval topology (4.11). Hence $\lambda_\Sigma X$ is a Kaufman orderable compactification (cf., 7.16).

PROPOSITION I(2). Every orderable compactification of X is a lattice superextension of X.

PROOF. Let Y be an orderable compactification of X. We will show that Y must be a topological lattice completion of X in which X is lattice dense. The desired conclusion will then follow from Theorem 6.20.

Y is a topological lattice completion of X: Since Y is compact in the interval topology, it is complete (2.9). Since X is contained in Y as both a subspace and a sublattice, it follows that the identity map $i: X \to Y$ is a continuous lattice inclusion. To see that $Y = X^2$, let $y \in Y$. It is easily seen that $z = \vee_Y \{x \in X: x \leq y\} \leq y \leq w = \wedge_Y \{x \in X: y \leq x\}$. Since Y is totally ordered, it follows that $X = i[X] \subseteq Y_D(z) \cup Y_I(w)$ which is a closed subset of Y, and since X is topologically dense in Y, y must be in $Y_D(z) \cup Y_I(w)$. Thus either $y = z$ or $y = w$. In either case we see that $y \in X^{1a} \cup X^{1b}$. Hence $Y \subseteq X^{1a} \cup X^{1b} \subseteq X^2 \subseteq Y$ so that $Y = X^2$.

X is lattice dense in Y: Suppose that $y \in X^{1a} - X^1$, $z \in X^{1b} - X^1$, and $z < y$. Thus there is some $x \in X$ with $x \leq y$ such that $x \neq z$. Since Y is totally ordered, it must be the case that $z < x < y$. Hence X is lattice dense in Y (cf. Definition 6.18).

APPENDIX II

Lattice Closures

The purpose of this Appendix is to show how Definition 6.1 can be extended to each ordinal in order to define the "lattice closure" of sets in a complete lattice and to show that this "lattice closure" operator is a Moore closure operator, or completion-operator as in Definition 5.1.

Throughout this Appendix, Y will denote a complete lattice and Z will denote a subset of Y. Note that if α is an ordinal, then α is an element of exactly one of the classes N, L, and M where $N = \{1,2,\cdots\}$, L is the class of limit ordinals, and $M = \{\alpha + n: \alpha \in L \text{ and } n \in N\}$. Keeping this in mind we state the following inductive definition:

NOTATION II(1). Suppose that $Z \neq \emptyset$.

(i) For each ordinal α, $\emptyset^{\alpha a} = \emptyset^{\alpha b} = \emptyset$.

(ii) Let Z^0, Z^{1a}, Z^{1b}, Z^1, Z^{2a}, Z^{2b}, and Z^2 be as defined in 6.1.

(iii) If $n \in N$ is even and we have defined Z^{na} and Z^{nb}, then we let

$$Z^{(n+1)a} = (Z^{na})^{1a} \text{ and } Z^{(n+1)b} = (Z^{nb})^{1b}.$$

If $n \in N$ is odd and we have defined Z^{na} and Z^{nb}, then we let

$$Z^{(n+1)a} = (Z^{na})^{1b} \text{ and } Z^{(n+1)b} = (Z^{nb})^{1a}.$$

(iv) If $\alpha \in L$ and we have defined $Z^{\beta a}$ and $Z^{\beta b}$ for each ordinal $\beta < \alpha$, then let

$$Z^{\alpha a} = \cup\{Z^{\beta a} : \beta < \alpha\} \text{ and } Z^{\alpha b} = \cup\{Z^{\beta b} : \beta < \alpha\}.$$

(v) If $\alpha \in L$ and we have defined $Z^{\alpha a}$ and $Z^{\alpha b}$, then let

$$Z^{(\alpha+1)a} = (Z^{\alpha a})^{1a} \text{ and } Z^{(\alpha+1)b} = (Z^{\alpha b})^{1b}.$$

(vi) If $\alpha + n \in M$, n is odd, and we have defined $Z^{(\alpha+n)a}$ and $Z^{(\alpha+n)b}$, then let

$$Z^{(\alpha+n+1)a} = (Z^{(\alpha+n)a})^{1b} \text{ and } Z^{(\alpha+n+1)1b} = (Z^{(\alpha+n)b})^{1a}.$$

If $\alpha + n \in M$, n is even, and we have defined $Z^{(\alpha+n)a}$ and $Z^{(\alpha+n)b}$, then let

$$Z^{(\alpha+n+1)a} = (Z^{(\alpha+n)a})^{1a} \text{ and } Z^{(\alpha+n+1)b} = (Z^{(\alpha+n)b})^{1b}.$$

(vii) For each ordinal α, let $Z^{\alpha} = Z^{\alpha a} \cap Z^{\alpha b}$.

PROPOSITION II(2). *For each ordinal* α, $Z^{\alpha} \subseteq Z^{\alpha a} \cup Z^{\alpha b} \subseteq Z^{\alpha+1}$.

By an argument similar to the one in the proof of Proposition 6.2 (iv) where it is shown that $Z^{1a} \cup Z^{1b} \subseteq Z^{2}$, the analogous conclusion is valid for each ordinal α.

COROLLARY II(3). *For each ordinal* α, $Z^{\alpha} \subseteq Z^{\hat{\alpha}}$ *for each ordinal* $\hat{\alpha}$ *such that* $\alpha < \hat{\alpha}$.

The proof is immediate from Proposition II(2) and Definition II(1) (iv) and (vii).

PROPOSITION II(4). *If* $Z \subseteq W \subseteq Y$, *then* $Z^{\alpha a} \subseteq W^{\alpha a}$, $Z^{\alpha b} \subseteq W^{\alpha b}$, *and* $Z^{\alpha} \subseteq W^{\alpha}$ *for each ordinal* α.

PROOF. If α is a non-limit ordinal, then the conclusion follows from an argument analogous to that in the proof of Proposition 6.2. If α is any limit ordinal and we have shown that $Z^{\beta a} \subseteq W^{\beta a}$ and $Z^{\beta b} \subseteq W^{\beta b}$ for each ordinal $\beta < \alpha$, then

$$Z^{\alpha a} = \cup\{Z^{\beta a} : \beta < \alpha\} \subseteq \cup\{W^{\beta a} : \beta < \alpha\} = W^{\alpha a} \text{ and}$$

$$Z^{\alpha b} = \cup\{Z^{\beta b} : \beta < \alpha\} \subseteq \cup\{W^{\beta b} : \beta < \alpha\} = W^{\alpha b};$$

therefore, $Z^{\alpha} = Z^{\alpha a} \cap Z^{\alpha b} \subseteq W^{\alpha a} \cap W^{\alpha b} = W^{\alpha}$.

PROPOSITION II(5). *There exists a least ordinal α such that* $Z^{\alpha} = Z^{(\alpha+1)}$.

PROOF. Since Y is a set, it has some cardinality. Hence the process of adding new elements of Y to Z must eventually stop; i.e., there is some ordinal α for which $Z^{\alpha} = Z^{\alpha+1}$. Since the class of ordinal numbers is well-ordered, there is some least ordinal α for which $Z^{\alpha} = Z^{\alpha+1}$.

In view of Proposition II(5), we may define the lattice closure of Z in Y unambiguously as follows:

DEFINITION II(6). If Z is a subset of Y, then the *lattice closure of Z in Y* is Z^{α} where α is the least ordinal such that $Z^{\alpha} = Z^{\alpha+1}$.

If Z is complete, then it follows that the lattice closure of Z in Y is $Z^{0} = Z$.

PROPOSITION II(7). The map $C: 2^Y \to 2^Y$ which takes each subset of Y onto its lattice closure in Y is a completion operator on Y.

PROOF. C1. Let $A \subseteq Y$. Then since $A = A^0$, it follows from Corollary II(3) that $A \subseteq C(A)$.

C2. Let $A \subseteq Y$. Then there is some least ordinal α such that $A^\alpha = A^{\alpha+1} = C(A)$. It follows from II(1) that $C(A)^1 = (A^\alpha)^1 = (A^{\alpha+1})^1 \subseteq A^{\alpha+2} = A^\alpha = C(A) = C(A)^0$ and hence $C(C(A)) = C(A)^0 = C(A)$.

C3. Suppose that $A \subseteq B \subseteq Y$. It follows from Proposition II(4) that $C(A) \subseteq C(B)$.

Thus the map C is a completion operator on Y (cf. Definition 5.1).

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BIOGRAPHICAL SKETCH

Linda Joe Wahl Smith was born May 8, 1944, in Jacksonville, Florida. In June, 1962, she graduated from Quincy High School, Quincy, Florida. In December, 1965, she received the degree of Bachelor of Science from Florida State University. In May, 1966, she entered the Graduate School of the University of Florida. She worked as a graduate assistant at the University Computing Center during the Spring Trimester 1965-66, and from September, 1966 through August, 1968 she was a National Science Foundation Trainee. In December, 1967, she received the degree of Master of Science from the University of Florida. From that time until the present she has worked toward the degree of Doctor of Philosophy. In September, 1968, she moved to Tallahassee, Florida, where she has worked as a part-time instructor of mathematics at Florida State University and Tallahassee Junior College.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December, 1969

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